TRAVELLING SALESMAN PROBLEM

MŰEGYETEM 1782

Dr. Tibor SIPOS Ph.D.

Dr. Árpád TÖRÖK Ph.D.

Zsombor SZABÓ

2019

BME FACULTY OF TRANSPORTATION ENGINEERING AND VEHICLE ENGINEERING

32708-2/2017/INTFIN COURSE MATERIAL SUPPORTED BY EMMI

Introduction

- Main reason to analyse the branch and bound technique is to solve the travelling salesman problem
- The travelling salesman problem (TSP) is one of the most important optimization problems
- The most famous problem for TSP is the Bridges of Königsberg

Problem

- Clay Mathematics Institute – Millenium Problems
- Seven most important problems which are unsolved
- $P = NP$ should be proven or negatived
- TSP is one of the most famous problem which NP solution (exponential timed) is known but the P solution (polinomial timed) is researched
- A US $1 million prize being awarded by the institute to the discoverer
- http://www.claymath.org/millennium-problems
- http://www.claymath.org/millennium-problems/p-vs-np-problem

Description

- In the travelling salesman problem there are n points of interests (for example cities), and the resistance between them
- The task is to plan the shortest route which includes all of the cities for exactly one time

Description

$$\min Z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

$$\sum_{j=1}^{n} x_{ij} = 1 \; \forall i$$

$$\sum_{i=1}^{m} x_{ij} = 1 \; \forall j$$

$$x_{ij} = \begin{cases} 0 \\ 1 \end{cases} \forall \, i, j$$

- Note, that there is another constraint which is difficult to formulate
- If the matrix is considered as a graph, then the chosen arcs must give one, and only one circle

Method

- Four steps:
  - Matrix reduction: Each element of each row are reduced by the row's minimum. Then this method is also used for the columns
  - On the zero cells the r values are counted
  - Choose the maximum of the r values
  - Then comes the branch and bound method. There are two possible ways. The chosen element can be eliminated or can be chosen into the route

Example

	A	B	C	D
A	M	6	2	4
B	1	M	2	7
C	4	8	M	5
D	5	2	6	M

Table form

- Note, that the elements in the matrix's main diagonal cannot be chosen, so they are signed by 'M'

Reducted matrix

	A	B	C	D
A	M	4	0	1
B	0	M	1	5
C	0	4	M	0
D	3	0	4	M

r values

- The second step is the calculating of the r values for all of the zeros

$$r_{\hat{i}\hat{j}} = \min_{j} c_{\hat{i}j} + \min_{i} c_{i\hat{j}}$$

- With this, the next r values are occurred

$$r_{AC} = 1 + 1 = 2$$
$$r_{BA} = 0 + 1 = 1$$
$$r_{CA} = 0 + 0 = 0$$
$$r_{CD} = 0 + 1 = 1$$
$$r_{DB} = 3 + 4 = 7$$

Branch and Bound

- The branch and bound method is need an initial bound to solve
- The initial bound for the problem is the sum of the rows' and columns' minimums during the matrix reduction ($Z^* = 10 = k_0$)
- In the branching method there are two branches, choosing or eliminating

Branch and Bound – Eliminating

- The branch's bound will be set

$$k_l = k_{l^-} + r_{ij}$$

- The eliminated cell must be signed by 'M', and then the matrix reduction will be the next step

Branch and Bound – Eliminating

	A	B	C	D
A	M	4	0	1
B	0	M	1	5
C	0	4	M	0
D	3	M	4	M

Branch and Bound – Choosing in

- If the recent element is chosen for the shortest route, then the cell itself ($c_{\hat{\imath}\hat{\jmath}}$), the whole row, the whole column and $c_{i=\hat{\jmath},j=\hat{\imath}}$ need to be set to 0
- The bound will not increase, only if a row or a column occur, where there will not be any zeros
- Because there is a 0 in every row and column, the k will not change

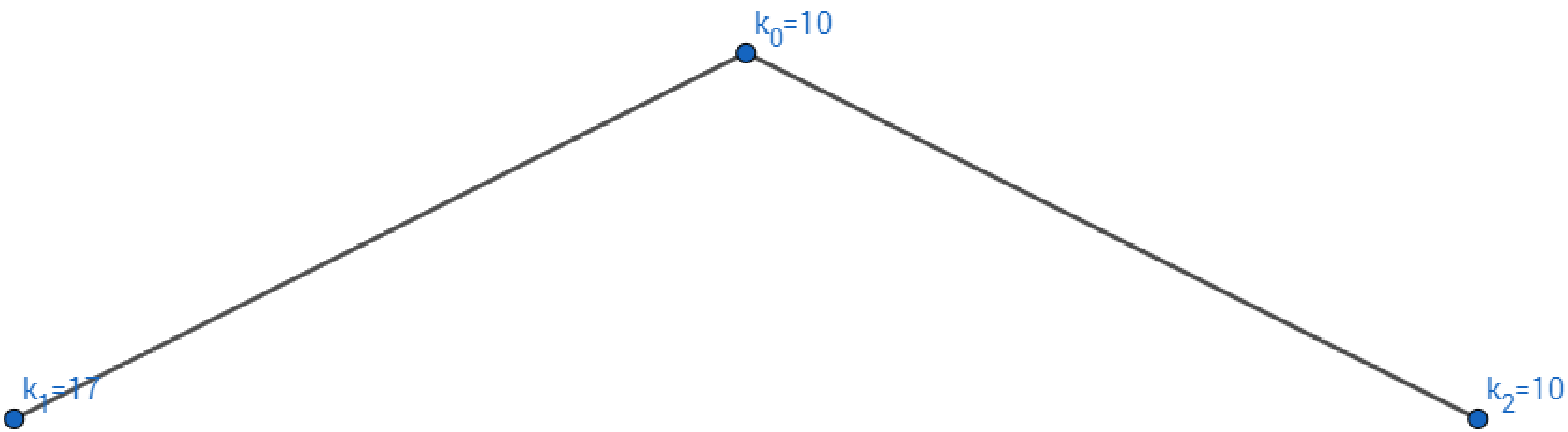
Branch and Bound – Choosing in

	A	B	C	D
A	M	M	0	1
B	0	M	1	M
C	0	M	M	0
D	M	M	M	M

Branch and Bound – Fathoming

- Some special issues are needed
- In these problems the Z^* will not occur, instead of this there is a recent k value
- The unfeasible subproblem is a remaining criteria
- During the fathoming step the subproblem with the lower bound is ought to be chosen

First Step – Enumeration Tree



Second Step – Initial Matrix

	A	B	C	D
A	M	M	0	1
B	0	M	1	M
C	0	M	M	0
D	M	M	M	M

Second Step – r values

- No more matrix reduction is needed
- r values are ought to be calculated

$$r_{AC} = 1 + 1 = 2$$
$$r_{BA} = 0 + 1 = 1$$
$$r_{CA} = 0 + 0 = 0$$
$$r_{CD} = 0 + 1 = 1$$

- AC cell should be used
- L_3: AC cell is eliminated ($k_3 = 12$)
- L_4: AC cell is used ($k_4 = 10$)

Second Step – Chosen element

	A	B	C	D
A	M	M	0	1
B	0	M	1	M
C	0	M	M	0
D	M	M	M	M

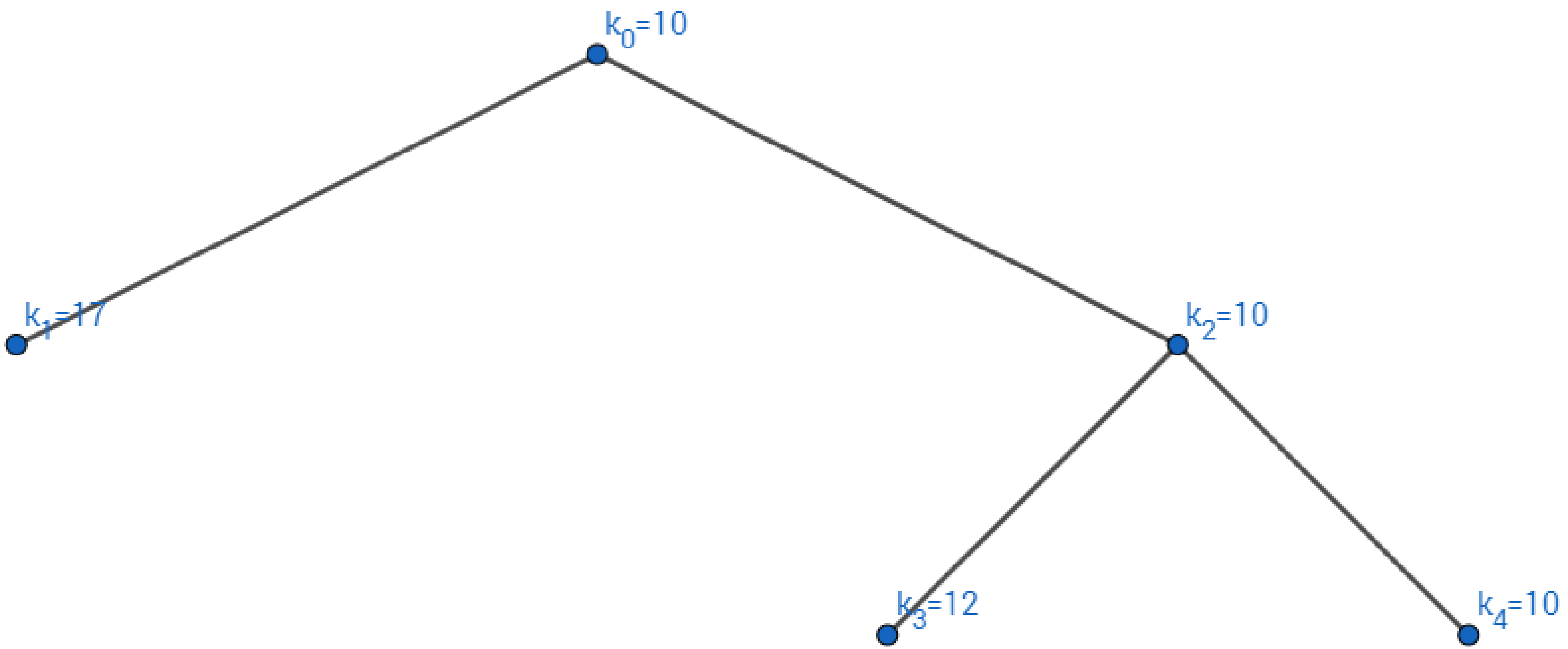
Second Step – Eliminating

	A	B	C	D
A	M	M	M	1
B	0	M	1	M
C	0	M	M	0
D	M	M	M	M

Second Step – Choosing in

	A	B	C	D
A	M	M	M	M
B	0	M	M	M
C	M	M	M	0
D	M	M	M	M

Second Step – Enumeration Tree



Third Step – Initial Matrix

	A	B	C	D
A	M	M	M	M
B	0	M	M	M
C	M	M	M	0
D	M	M	M	M

Third Step – Conclusions

- There are only two feasible cells remained, which are our missing sections
- So the optimal solution is occurred:

$$D \rightarrow B \rightarrow A \rightarrow C \rightarrow D$$

- Because the new actual optimal solution is $k_4 = 10$ then the higher branches should be fathomed

Optimal solution: $D \rightarrow B \rightarrow A \rightarrow C \rightarrow D$

	A	B	C	D
A	M	6	2	4
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BUDAPEST UNIVERSITY OF TECHNOLOGY AND ECONOMICS

Dr. Tibor SIPOS Ph.D.
Dr. Árpád TÖRÖK Ph.D.
Zsombor SZABÓ

email: szabo.zsombor@mail.bme.hu

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