NETWORK OPTIMISATION

MŰEGYETEM 1782

Dr. Tibor SIPOS Ph.D.

Dr. Árpád TÖRÖK Ph.D.

Zsombor SZABÓ

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Content

Graph Theory

Shortest Route Problem

Maximum Flow Problem

Minimum Cost Flow Problem

Graph theory

- Network: bunch of nodes and arcs
- Node: where the arcs intersect each other
- Arc: the nodes are connected by arcs
- In transportation sciences, the networks often have some kind of flow on the arcs
- Directed arc: the flow on the arc can be in only one direction
- Directed network: all of the arcs are directed

Graph theory

- Path: a bunch of arcs between two nodes
- Undirected path: all of the arcs in the path is undirected
- Circle: the path beginning and ending is the same node
- Connected nodes: there is an undirected path between the two nodes
- Connected network: all pair of nodes are connected to each other
- Tree: connected network without cycles
- Spanning tree: all of the nodes are in the tree

Content

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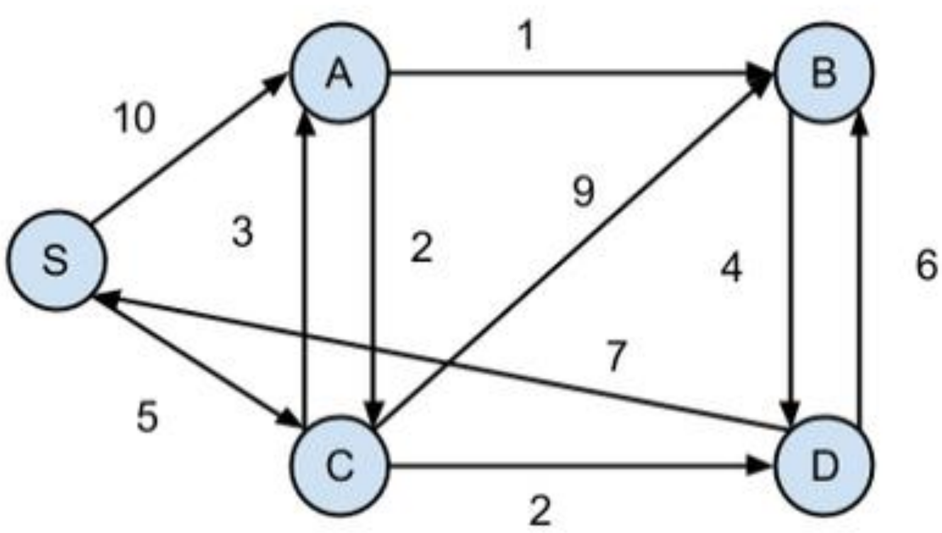
Dijkstra method

- Goal: find a minimal spanning tree, where there is a starting point chosen arbitrary, and the other nodes distances are calculated from this point
- Theorem: a spanning tree with n nodes has always $n-1$ arcs
- All of the networks can describe in graph and table form also

Dijkstra method



	A	B	C	D	S
A	M	1	2	M	M
B	M	M	M	4	M
C	3	9	M	2	M
D	M	6	M	M	7
S	10	M	5	M	M

Dijkstra method

- The table has three section
  - In the first column there are the chosen nodes, as they have been chosen
  - The second section ($n-1$ columns) shows the shortest path from the origin node
  - The third section, also $n-1$ columns, show the node before the destination node
- This can be made from node-to-node, row-to-row, step-to-step
- When an alternate route become in sight, which is shorter than the route that already picked, the new route must be used
- The method has always $n-1$ steps to solve

	A	B	D	S	A	B	D	S
C	3	9	2		C	C	C	
CD	3	8	2	9	C	D	C	D
CDA	3	4	2	9	C	A	C	D
CDAB	3	4	2	9	C	A	C	D

Dijkstra method

- Consider all of the nodes which can be reached from the chosen nodes, and calculate the shortest path to them
- If the path's length is shorter than the already counted, then the shorter path's length must be used in the next

Dijkstra method

	A	B	D	S	A	B	D	S
C	3	9	2		C	C	C	

Dijkstra method

	A	B	D	S	A	B	D	S
C	3	9	2		C	C	C	
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Dijkstra method

	A	B	D	S	A	B	D	S
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Dijkstra method

	A	B	D	S	A	B	D	S
C	3	9	2		C	C	C	
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CDAB	3	4	2	9	C	A	C	D

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Maximum flow problem

- Given a network
- Source: from the flows start
- Sink: flows’ destination
- Task: the most possible flows have been transported from the source to the sink

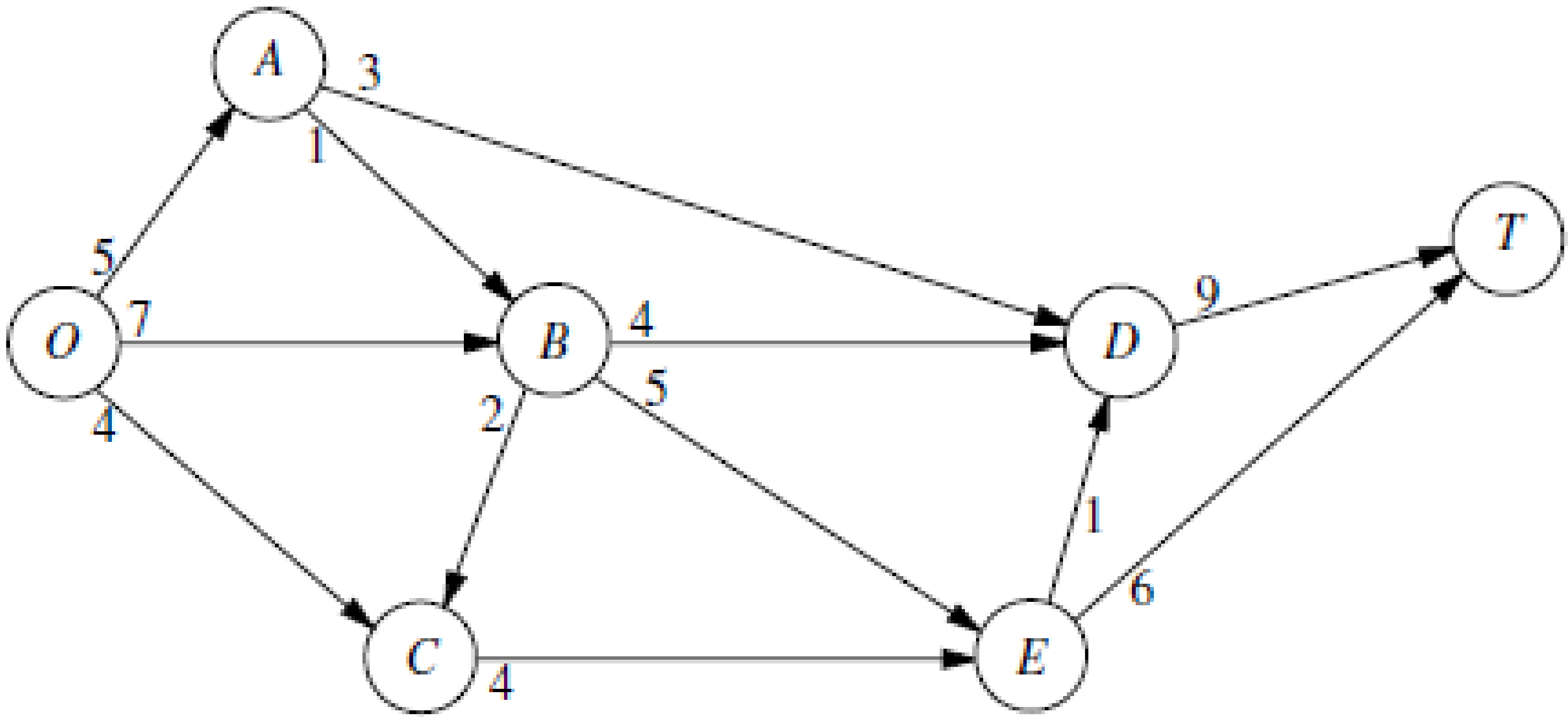
Ford-Fulkerson method

- n nodes
- Between the nodes, there are directed or undirected arcs
- Each of the arcs have an origin (i), and a destination (j), where $i, j \in [1..n]$
- All of the arcs also have a capacity (u_{ij}), and a flow (x_{ij}) with a constraint

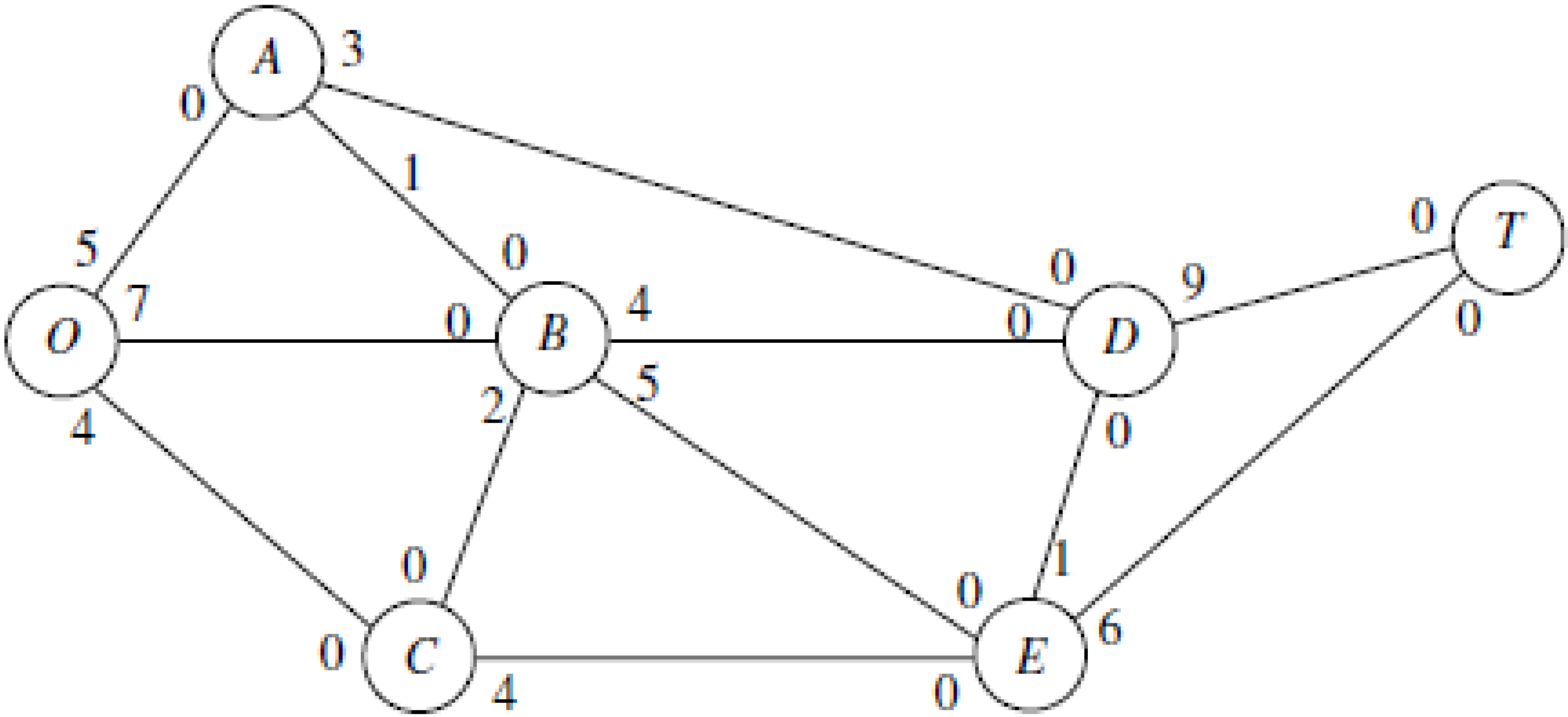
Ford-Fulkerson method

- Ford-Fulkerson method has two main steps
  - Find an augmenting path
    - Augmenting path: bunch of directed arcs, which residual capacities $(u_{ij} - x_{ij})$ are more than zero
  - If an augmenting path is found, then a flow, which is equal to the minimum of the remaining capacities, must be programmed onto the network
- Example:
  - On the network, the source is the node O, and the sink is the node T
  - All of the arcs have a direction, and a capacity written on it

Ford-Fulkerson method



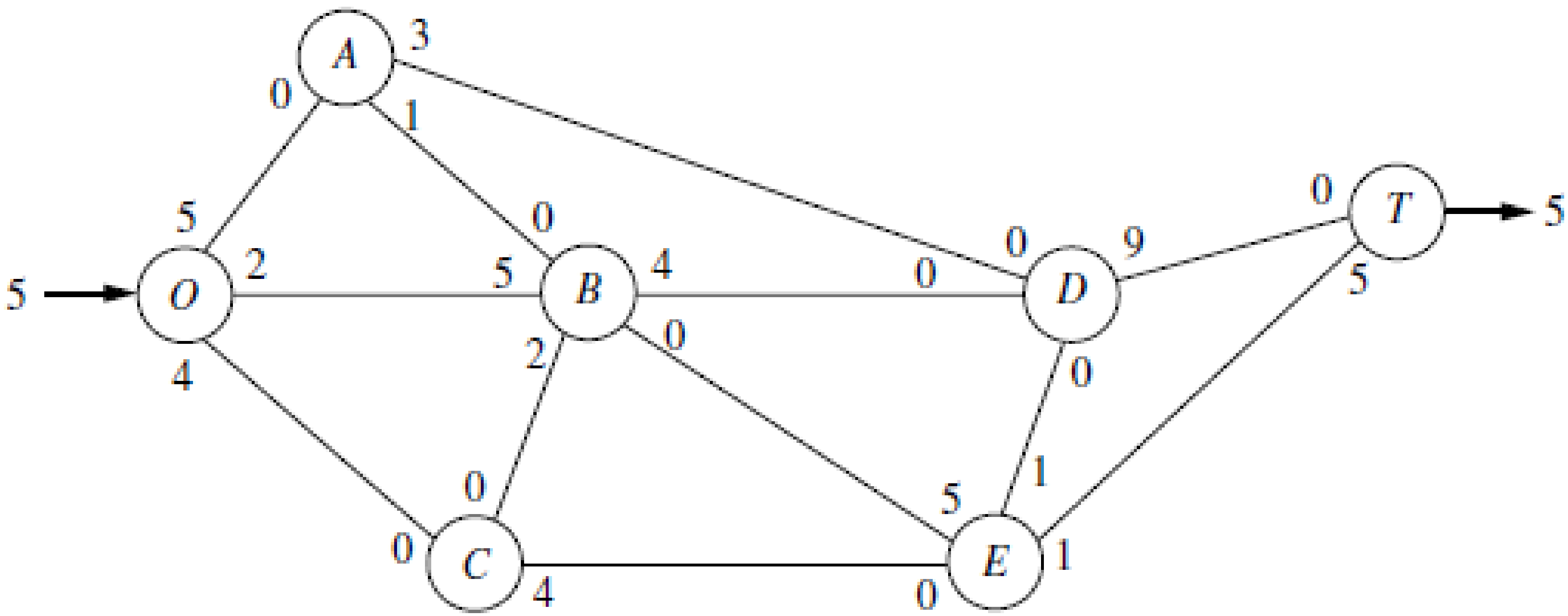
Ford-Fulkerson method



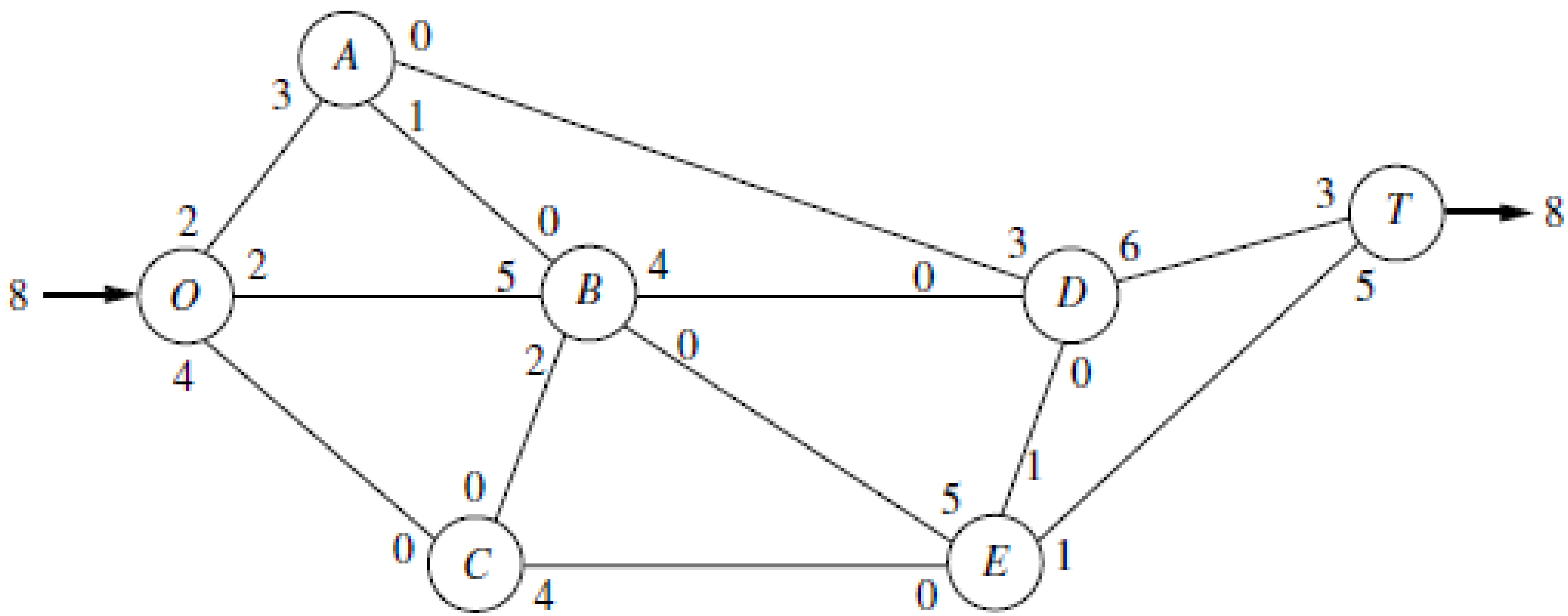
Ford-Fulkerson method

- All of the arcs have two numbers
- The first shows the residual capacity, and the second shows the actual flow on it
- For example, take into consideration the $O \to B \to E \to T$ augmenting path
  - The minimum residual capacity is the 5 (on the BE arc) so it can be programmed on the network
  - The programmed flow must be reduce from the involved arcs first number, and have to add to the second

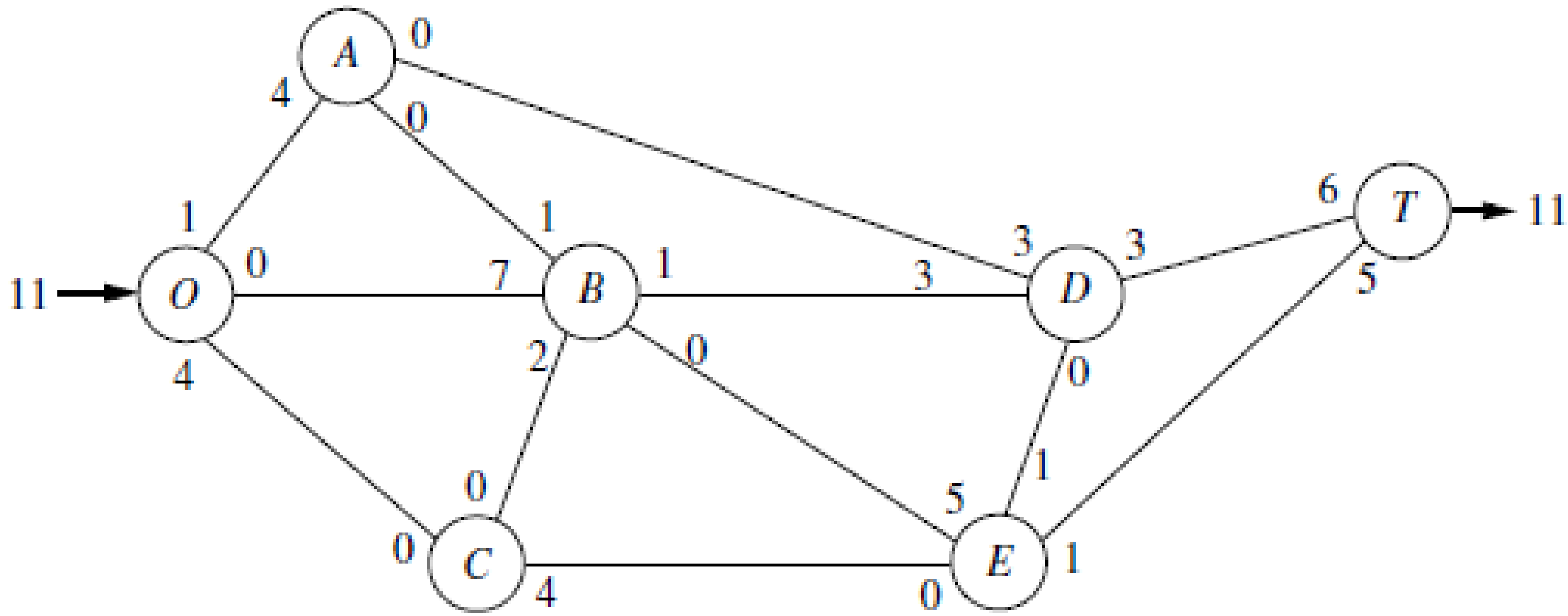
Ford-Fulkerson method: $O \to B \to E \to T$



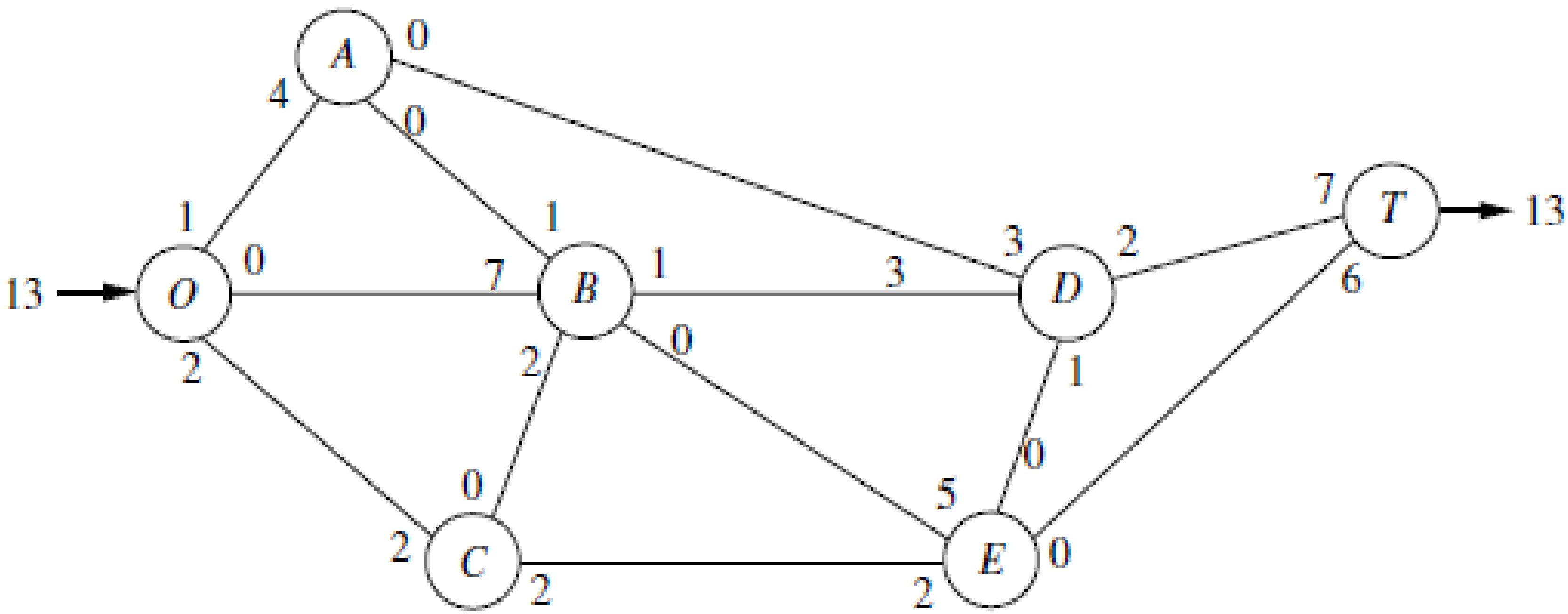
Ford-Fulkerson method: $O \rightarrow A \rightarrow D \rightarrow T$



Ford-Fulkerson method: $O \rightarrow A \rightarrow B \rightarrow D \rightarrow T$
$O \rightarrow B \rightarrow D \rightarrow T$



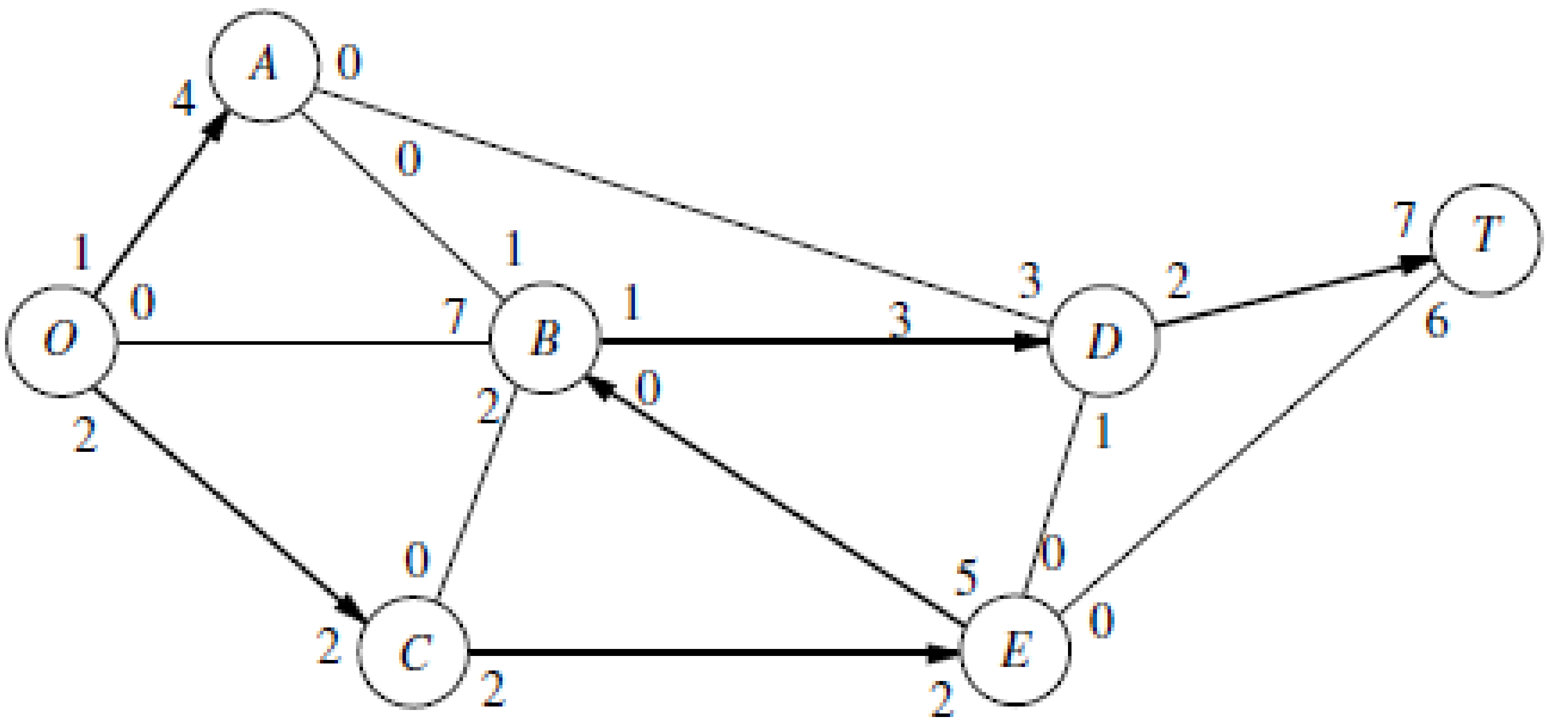
Ford-Fulkerson method: $O \rightarrow C \rightarrow E \rightarrow D \rightarrow T$
$O \rightarrow C \rightarrow E \rightarrow T$



Ford-Fulkerson method

- Important question is, that how an augmenting path can be found
- The method is that we start from the source and sign all of the nodes where the arc's first number (which is closer to the already signed node) is more than zero

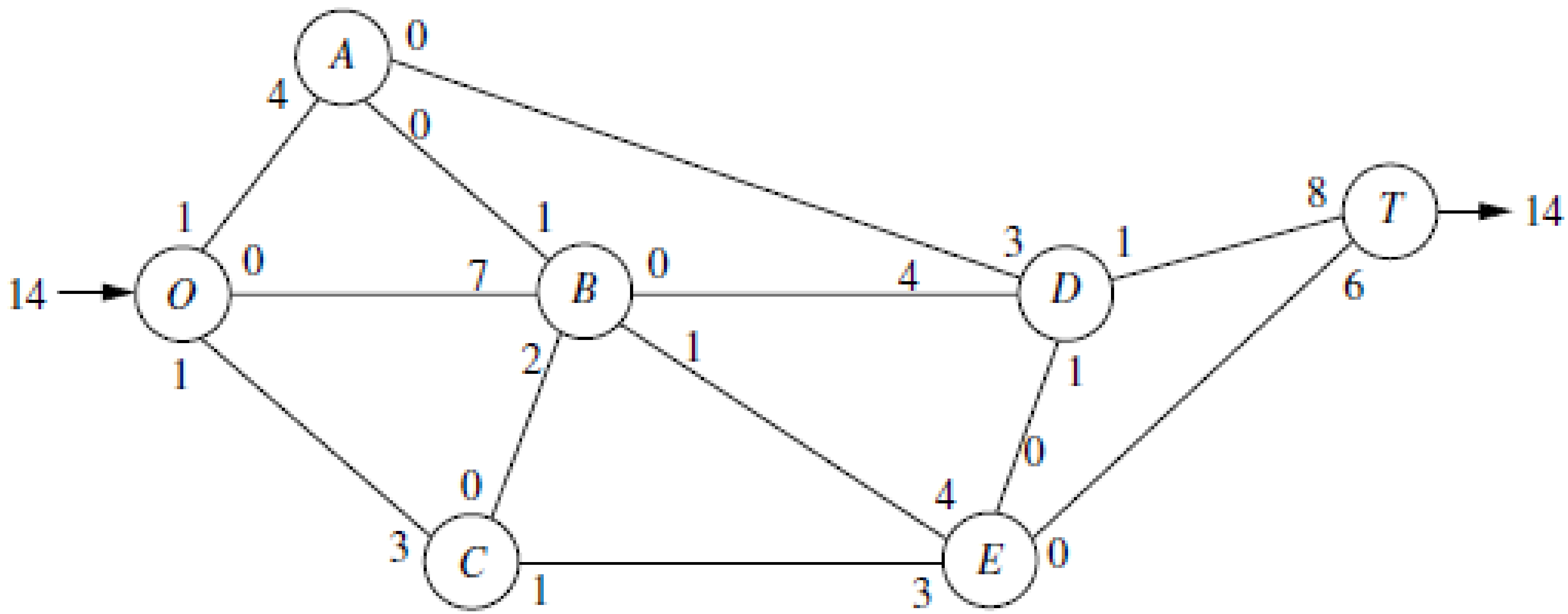
Ford-Fulkerson method



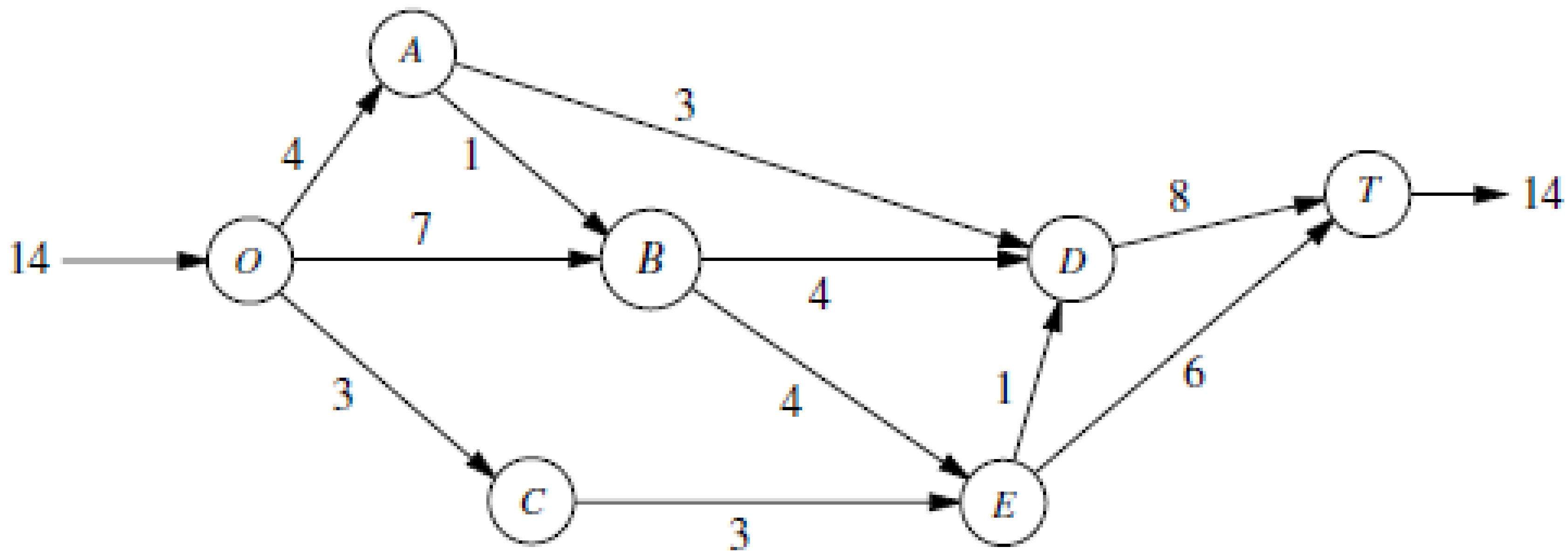
Ford-Fulkerson method

- Note, that in this case the BE arc's direction is opposite
- Some unit of programmed flow can be unprogrammed, and the connecting flows must be redirected to other route
- $O \rightarrow C \rightarrow E \rightarrow B \rightarrow D \rightarrow T$ route can be found

Ford-Fulkerson method



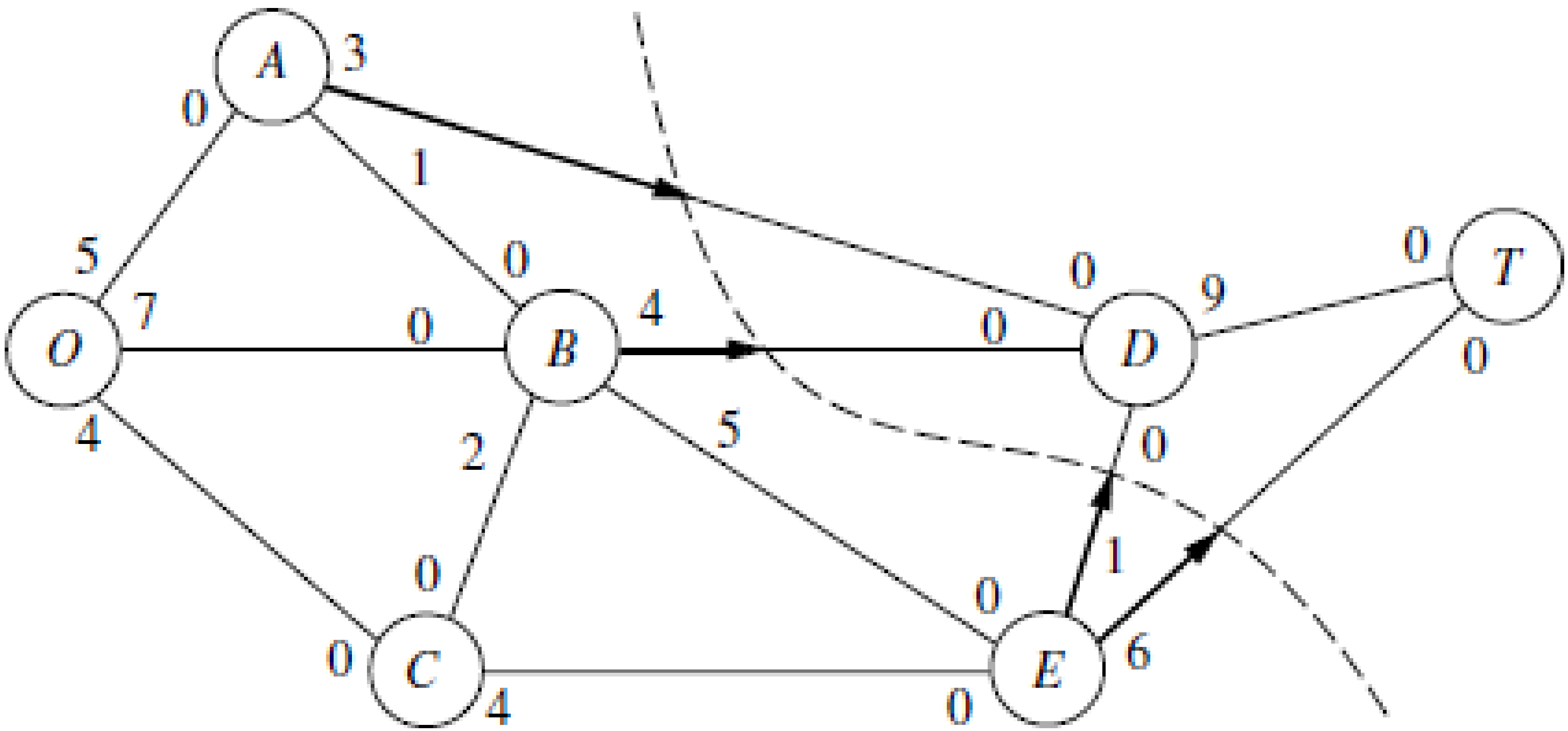
Ford-Fulkerson method



Ford-Fulkerson method – Optimality test

- Finding an augmenting path can be difficult in huge networks
- Optimality test: max-flow min-cut theorem
- Cut: any set of directed arcs containing at least one arc from every directed path from the source to the sink
- Cut value: the sum of the arc capacities of the arcs (in the specified direction) of the cut
- **Theorem:** max-flow min-cut: for any network with a single source and sink, the maximum feasible flow from the source to the sink equals the minimum cut value for all cuts of the network

Ford-Fulkerson method – Optimality test



Content

Graph Theory

Shortest Route Problem

Maximum Flow Problem

Minimum Cost Flow Problem

Content – Minimum Cost Flow Problem

Introduction

How to Transform the Networks into Minimum Cost Flow Problem

Network Simplex

Minimum cost flow problem

- The minimum cost flow problem holds a central position among network optimization models, both because it encompasses such a broad class of applications and because it can be solved extremely efficiently
- **All of the previously shown problems are special cases of this problem**

Minimum cost flow problem

- Have to set up two matrices
  - $\boldsymbol{X}$: contains the flows, which means that the flow between i and j nodes is x_{ij}
  - $\boldsymbol{C}$: contains the costs of the unit flows between i and j (c_{ij})
- All of the arcs in the network have a capacity (u_{ij}) which means the maximum feasible flow on the arcs
- And each node has a number (b_i), which mean the net flow generated at node i
  - If $b_i > 0$, then the node is called supply node (source)
  - If $b_i < 0$, then the node is demand node (sink)

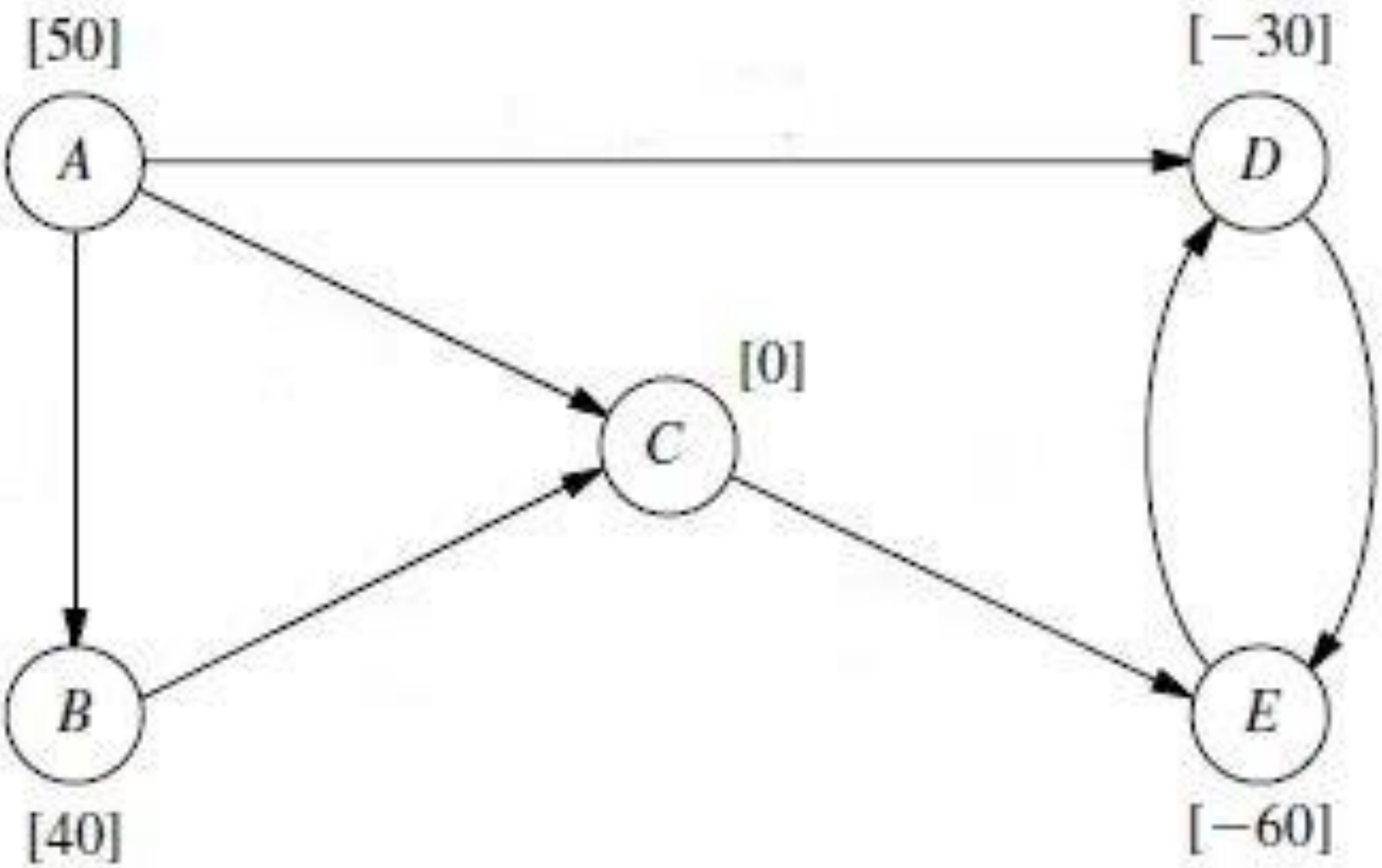
Minimum cost flow problem

$$\min Z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}$$

$$\sum_{j=1}^{n} x_{ij} - \sum_{j=1}^{n} x_{ji} = b_i \;\forall i$$

$0 \leq x_{ij} \leq u_{ij}$ for each arc

Network simplex method



Content – Minimum Cost Flow Problem

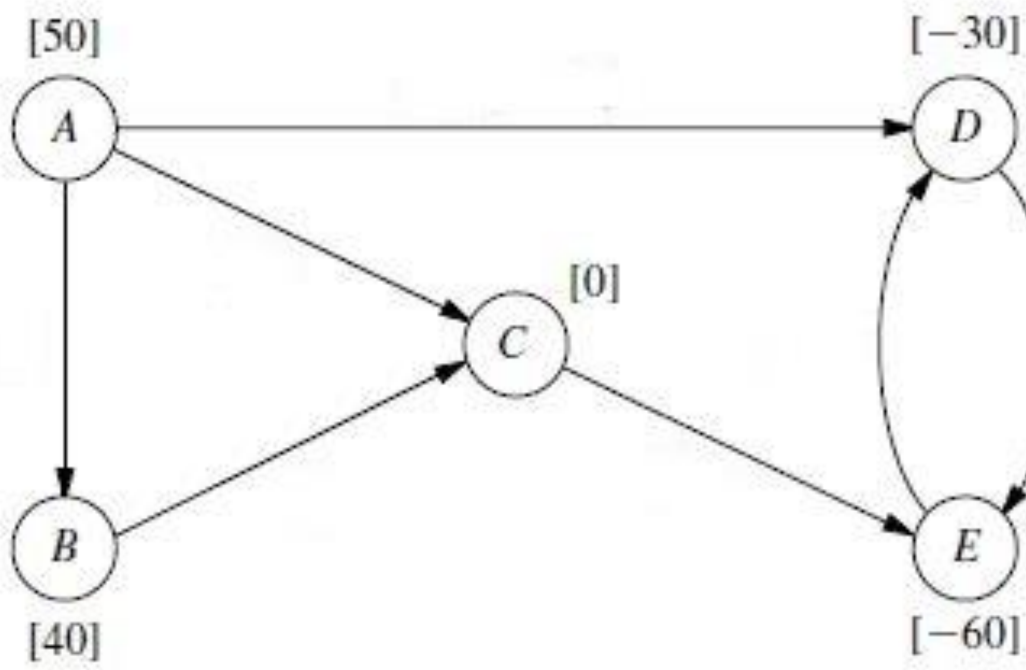
Introduction

How to Transform the Networks into Minimum Cost Flow Problem

Network Simplex

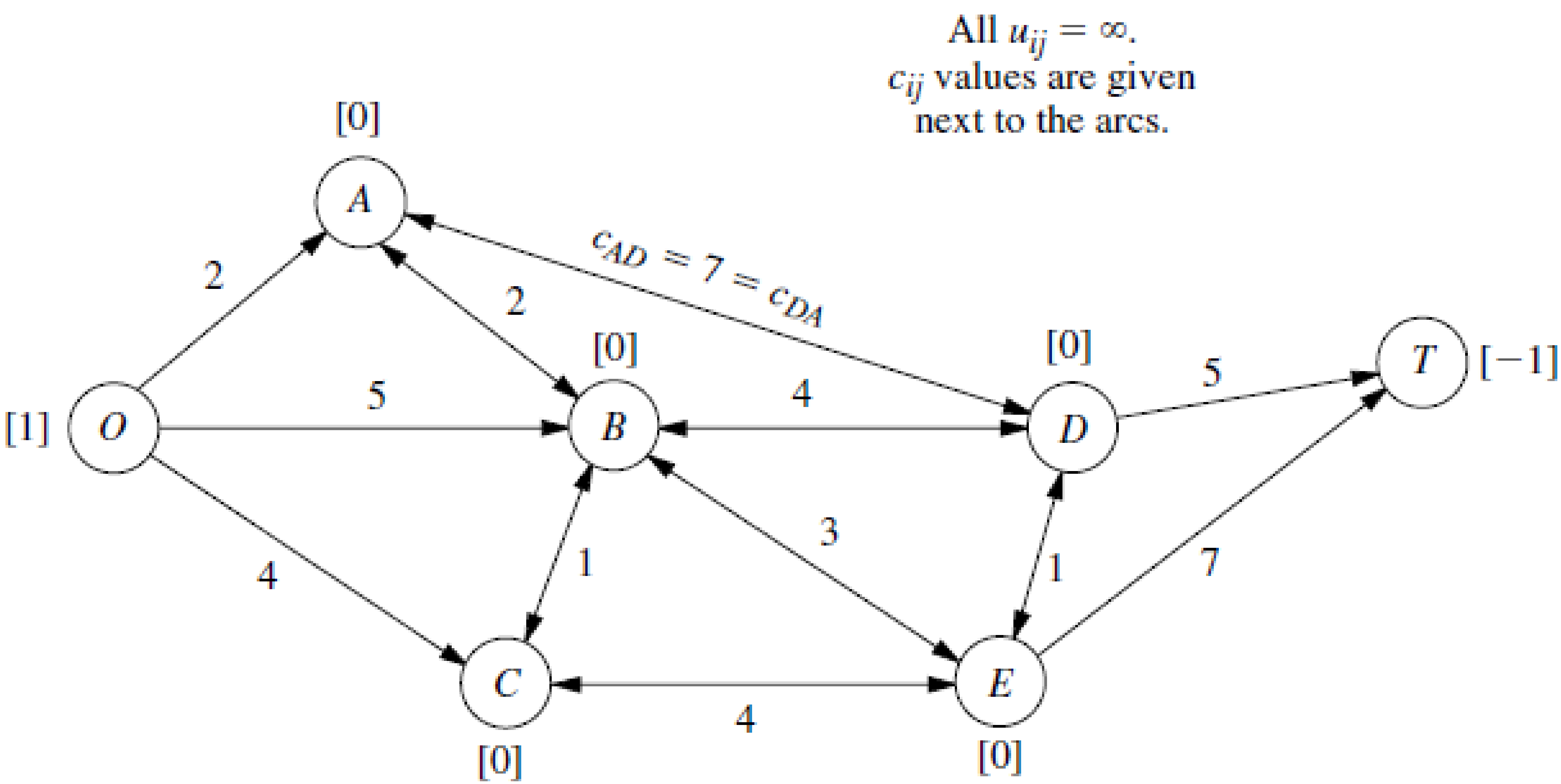
Transportation problem

- **Theorem:** Every minimum-cost flow problem with finite capacities or nonnegative costs has an equivalent transportation problem

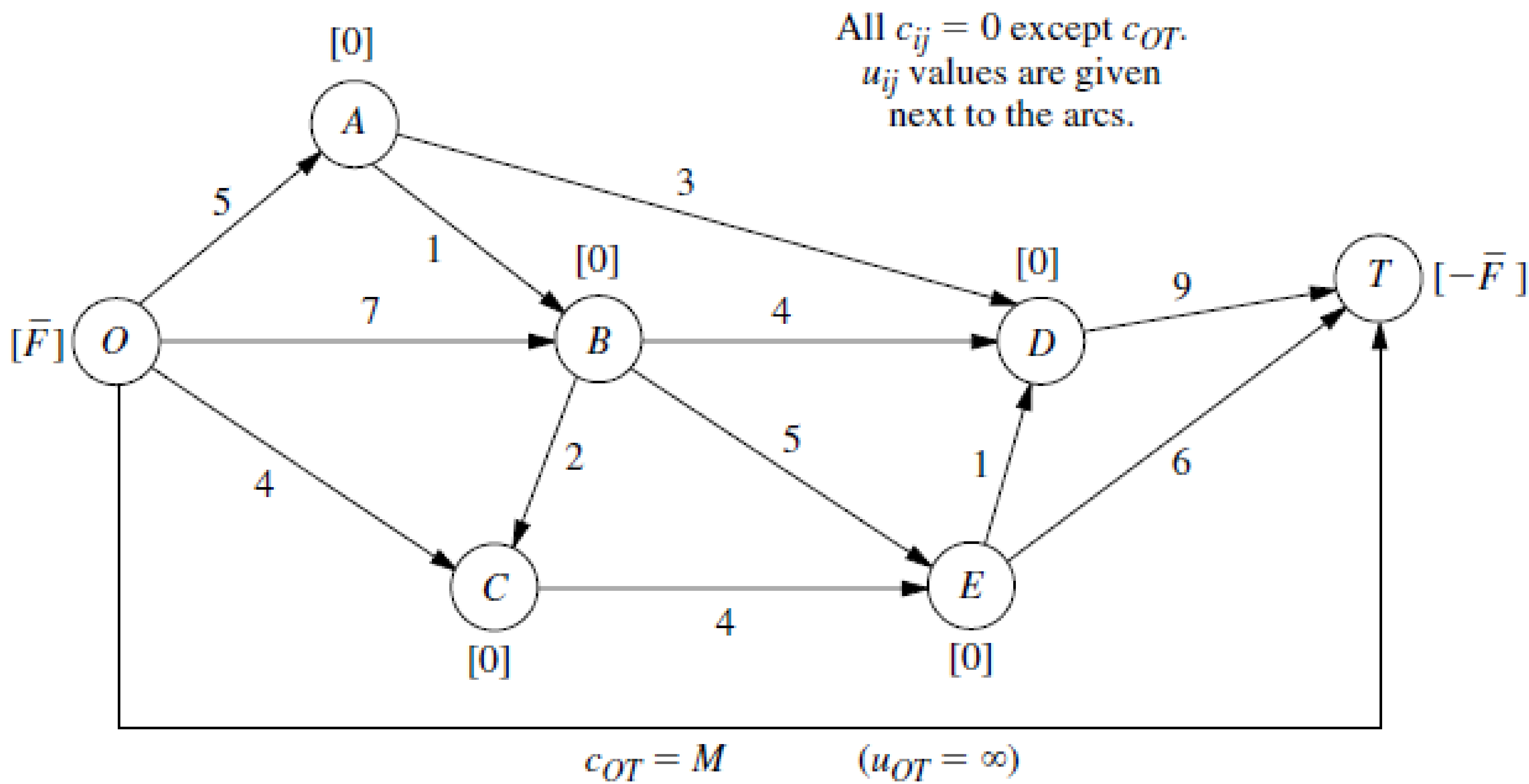


	D	E	
A			50
B			40
	30	60	

Shortest route problem



Maximum flow problem



Content – Minimum Cost Flow Problem

Introduction

How to Transform the Networks into Minimum Cost Flow Problem

Network Simplex

Network simplex method

- Has two main steps
  - Finding the leaving arc
  - Finding the next basic feasible (BF) solution
- All of the feasible solutions are spanning tree solutions
- Feasible spanning tree: a spanning tree whose solution from the node constraints also satisfies all the other constraints ($0 \leq x_{ij} \leq u_{ij}$ or $0 \leq y_{ij} \leq u_{ij}$)
- **Theorem:** fundamental theorem for the network simplex method: basic solutions are spanning tree solutions (and conversely) and that BF solutions are solutions for feasible spanning trees (and conversely)

Network simplex method

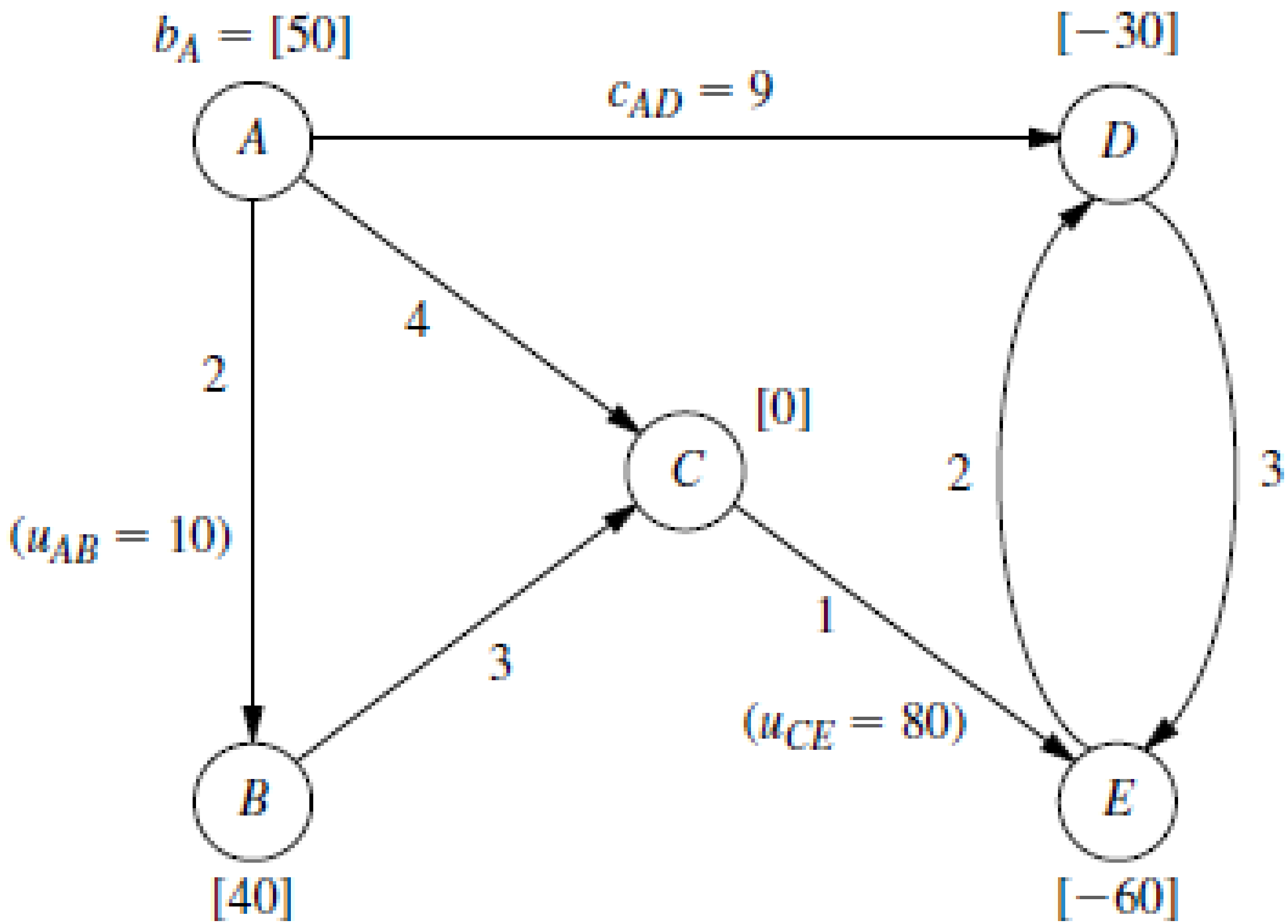
- Two arcs have constraints on it
  - $u_{AB} = 10$
  - $u_{CE} = 80$
- The cost of the arcs are the next

c_{ij}	A	B	C	D	E
A		2	4	9	
B			3		
C					1
D					3
E				2	

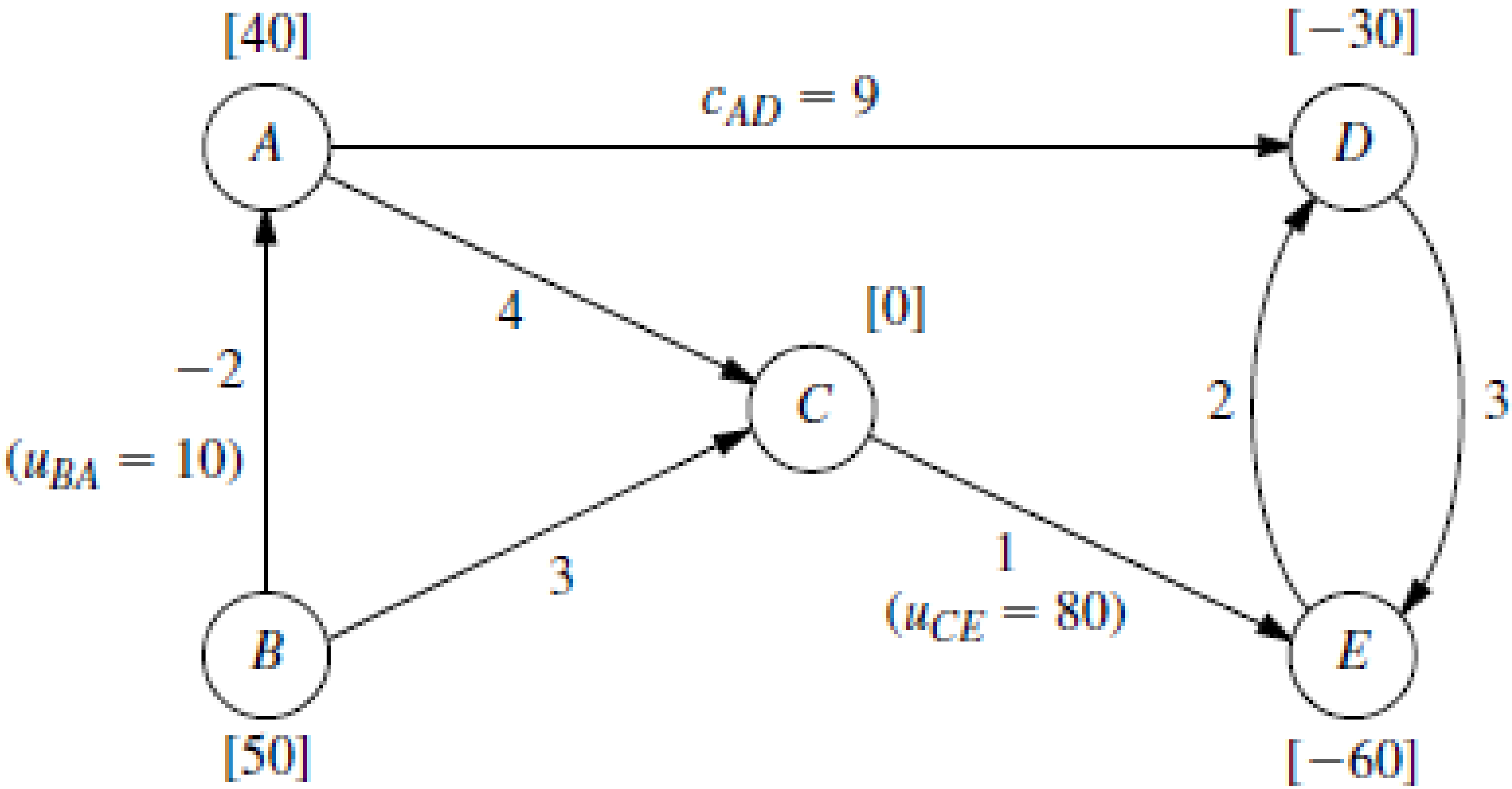
Upper bound technique

- For demonstrate the upper bound technique, program 10 units of flow onto the AB arc
- The direction of the AB arc must be changed
- $c_{BA} = -2$ must be set
- Node A's and node B's supplies must increase and decrease by 10
- The redirected arc must be treated in other way, so the decision variable of it must change to y_{BA}
- If some units of flow will be programmed on this arc in the next, that means that we decrease the flow on this arc

Upper bound technique



Upper bound technique

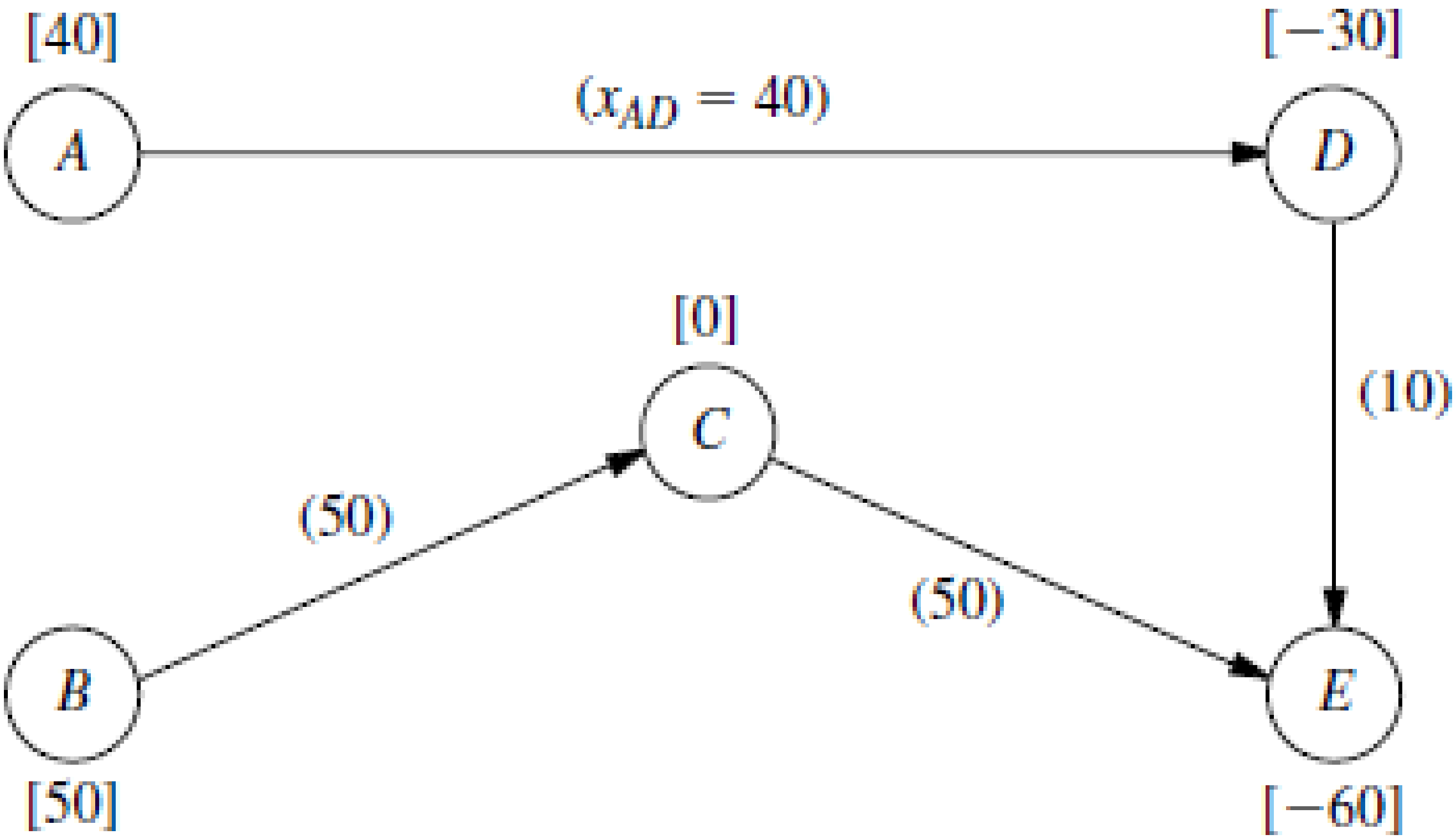


Initial feasible solution

- An initial feasible solution is need to be set up

$$
\begin{array}{lllllllll}
 & & & & & & & y_{AB}=0 \quad x_{AC}=0 \quad x_{ED}=0 & \\
\hline
-y_{AB} & +x_{AC} & +x_{AD} & & & & =40 & \text{so} & x_{AD}=40 \\
y_{AB} & & & +x_{BC} & & & =50 & \text{so} & x_{BC}=50 \\
 & -x_{AC} & & -x_{BC} & +x_{CE} & & =0 & \text{so} & x_{CE}=50 \\
 & & -x_{AD} & & & +x_{DE} -x_{ED} & =-30 & \text{so} & x_{DE}=10 \\
 & & & & -x_{CE} & -x_{DE} +x_{ED} & =-60 & \text{redundant} &
\end{array}
$$

Initial feasible solution



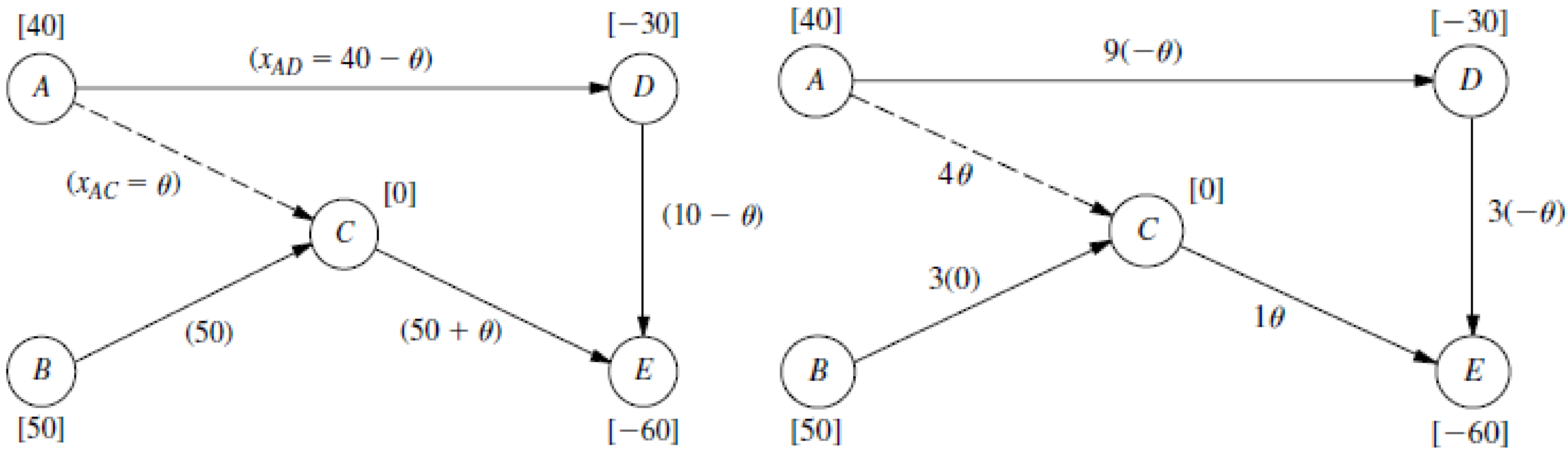
Selecting the Entering Basic Variable

- To begin an iteration of the network simplex method, recall that the standard simplex method criterion for selecting the entering basic variable is to choose the nonbasic variable which, when increased from zero, will improve Z at the fastest rate
- **Kirchhoff I. law:** junction rule: The algebraic sum of currents in a network of conductors meeting at a point is zero
- **Kirchhoff II. law:** loop rule: The directed sum of the electrical potential differences (in this recent case: the flows) around any closed network is zero

Selecting the Entering Basic Variable

- To choose the entering basic variable, take into consideration all of the nonbasic arcs
- Program on each actual arc a θ units of flow
- Then increase or decrease all of the basic arc's flow as Kirchhoff's second law said
- Then Z must be analysed

Selecting the Entering Basic Variable – for AC arc



$$\Delta Z = c_{AC}\theta + c_{CE}\theta - c_{DE}\theta - c_{AD}\theta = 4\theta + \theta - 3\theta - 9\theta = -7\theta$$

$$\Delta Z = \begin{cases} -7 \text{ if } \Delta x_{AC} = 1 \\ 6 \text{ if } \Delta y_{AB} = 1 \\ 5 \text{ if } \Delta x_{ED} = 1 \end{cases}$$

Finding the Leaving Basic Variable

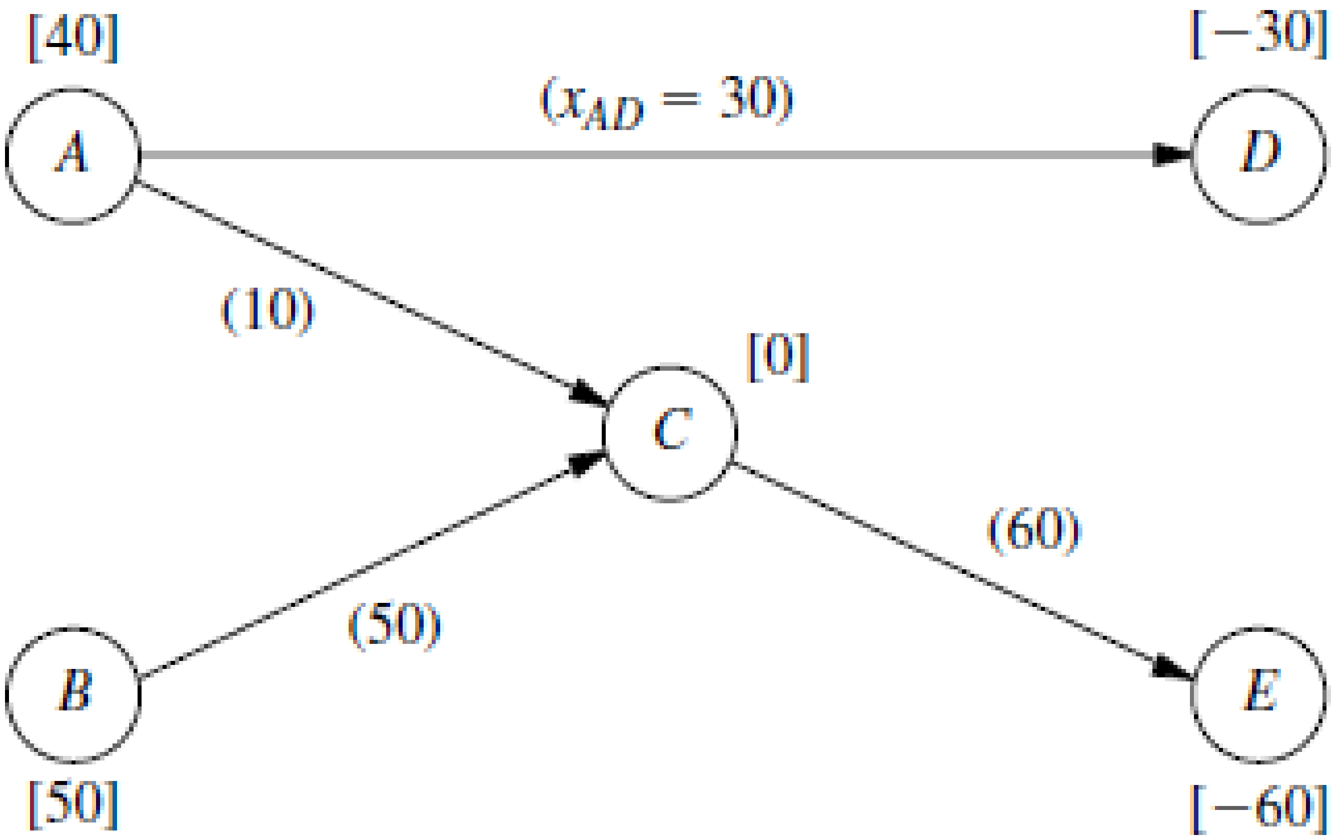
- For finding the leaving basic variable, the task is to find the maximum value of the θ, where all of the arc's constraints are satisfied
- When the flows are decreasing this bound is the nonnegativity constraint, but when the flows are decreasing they are the capacity bounds

Finding the Leaving Basic Variable

$$x_{AC} = \theta \leq \infty$$
$$x_{CE} = 50 + \theta \leq 80, \text{so } \theta \leq 30$$
$$x_{DE} = 10 - \theta \geq 0, \text{so } \theta \leq 10$$
$$x_{AD} = 40 - \theta \geq 0, \text{so } \theta \leq 40$$

- To satisfy all of the constraints the $\theta = 10$ will be the best choice
- A nonnegativity constraint will be the bound, so the use of the upper bound technique will not happen in this step

Actual Solution after the First Step



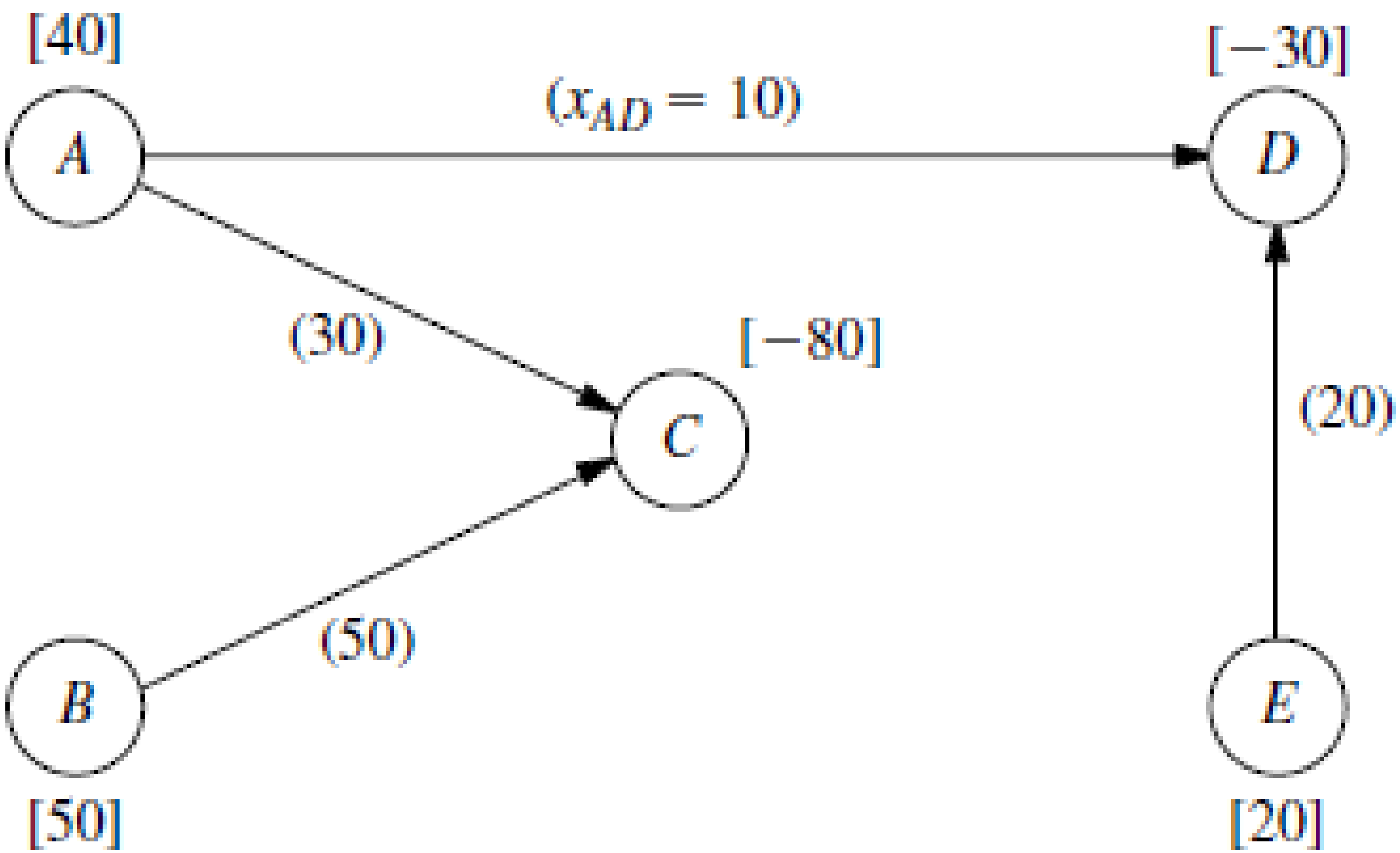
2nd step

$$\Delta Z = \begin{cases} 7 \text{ if } \Delta x_{DE} = 1 \\ -1 \text{ if } \Delta y_{AB} = 1 \\ -2 \text{ if } \Delta x_{ED} = 1 \end{cases}$$

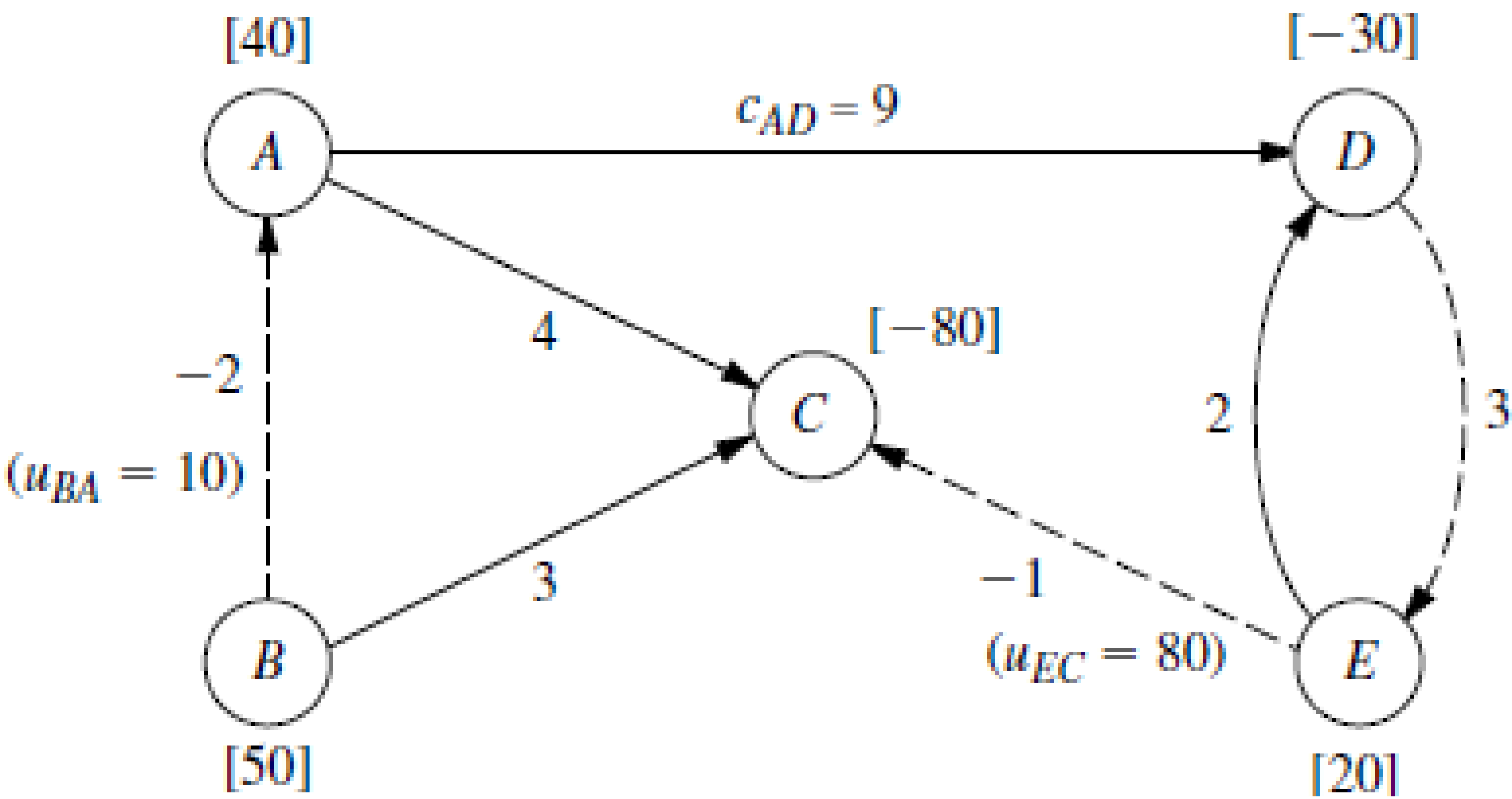
$$x_{ED} = \theta \leq \infty, \text{so } \theta \leq \infty$$
$$x_{AD} = 30 - \theta \geq 0, \text{so } \theta \leq 30$$
$$x_{AC} = 10 + \theta \leq \infty, \text{so } \theta \leq \infty$$
$$x_{CE} = 60 + \theta \leq 0, \text{so } \theta \leq 20$$

- Because upper bound is reached, upper bound technique is needed to use

2nd step – Results



2nd step – Results



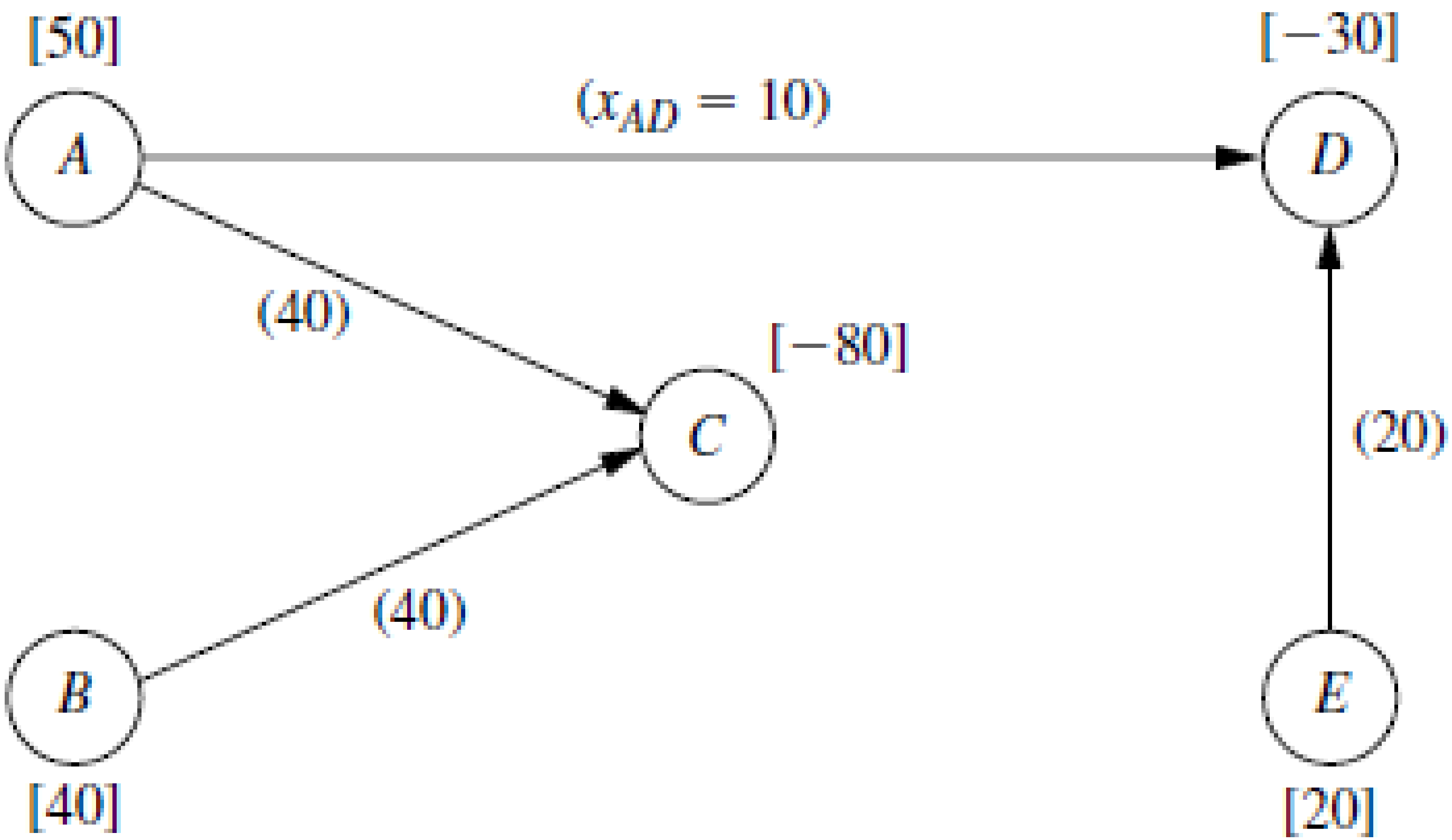
3rd step

$$\Delta Z = \begin{cases} 5 \text{ if } \Delta x_{DE} = 1 \\ -1 \text{ if } \Delta y_{AB} = 1 \\ 2 \text{ if } \Delta y_{EC} = 1 \end{cases}$$

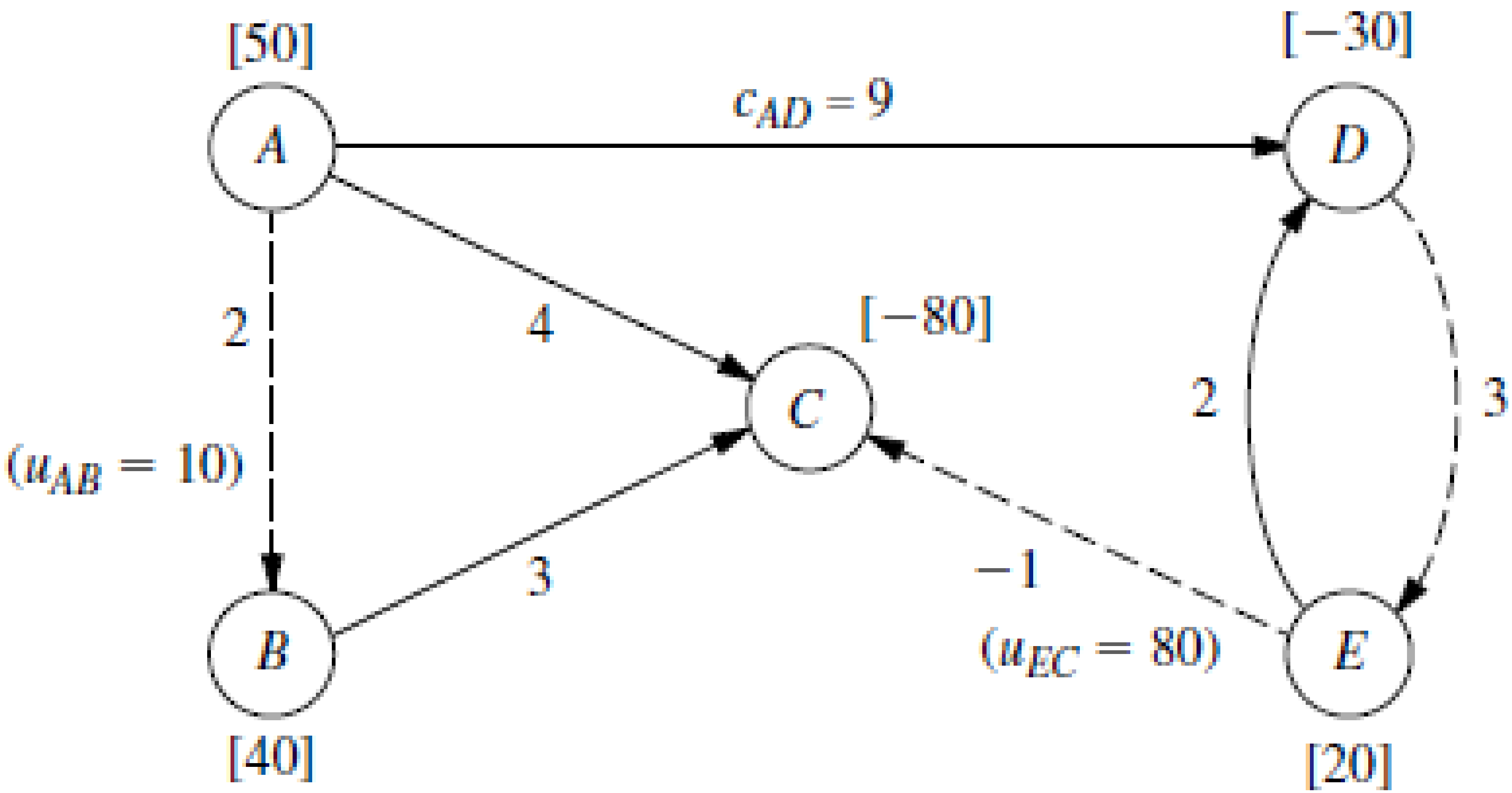
$$y_{AB} = \theta \leq 10, \text{so } \theta \leq 10$$
$$x_{AC} = 30 + \theta, \text{so } \theta \leq \infty$$
$$x_{BC} = 50 - \theta \geq 0, \text{so } \theta \leq 50$$

- Because upper bound is reached, upper bound technique is needed to use

3rd step – Results



3rd step – Results

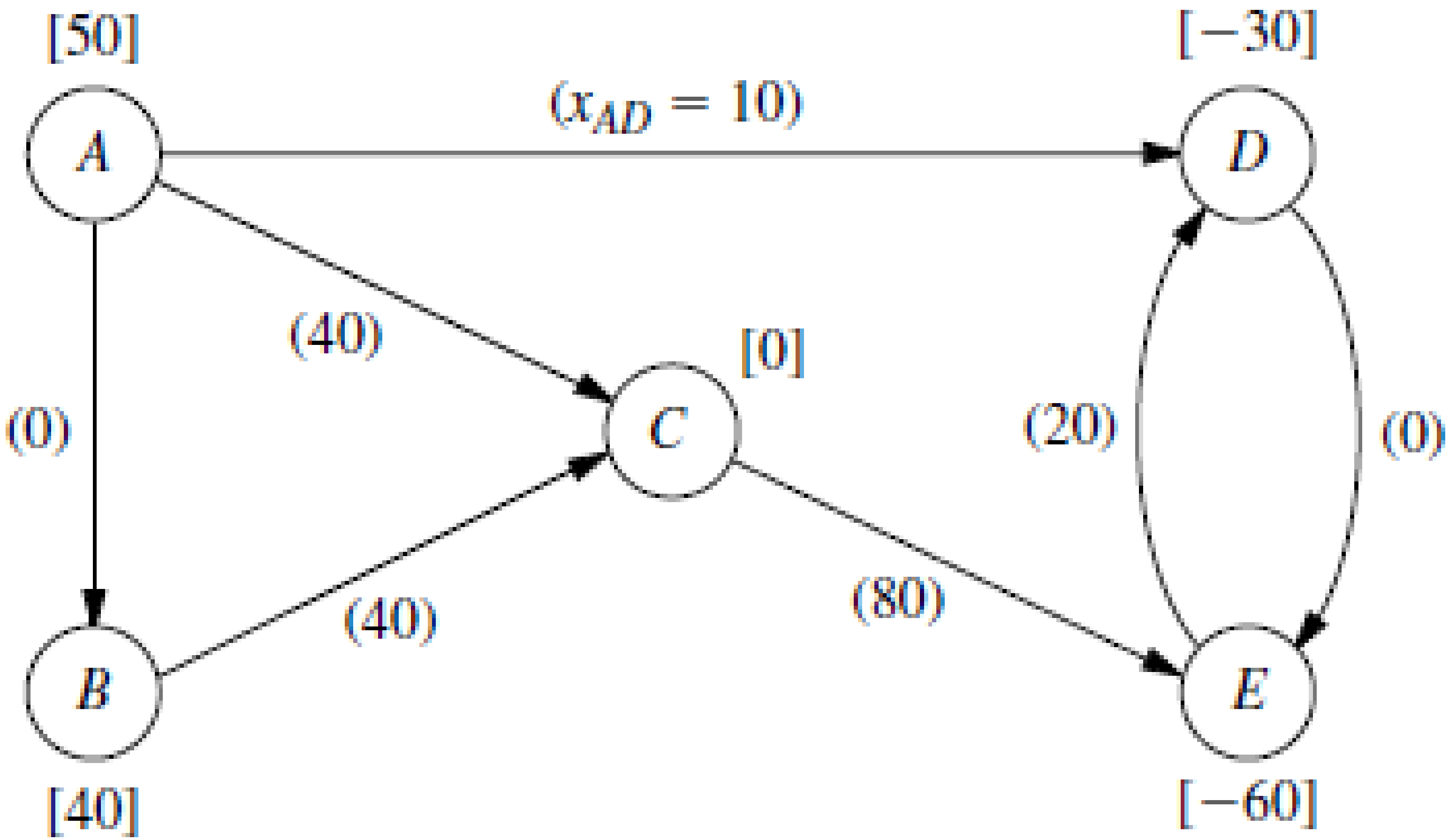


4th step

$$\Delta Z = \begin{cases} 1 \text{ if } \Delta x_{AB} = 1 \\ 7 \text{ if } \Delta x_{DE} = 1 \\ 2 \text{ if } \Delta y_{EC} = 1 \end{cases}$$

- None of the new arcs decreasing the objective function
- Optimal solution is reached

Optimal solution



Conclusion

<table>
<tr><th colspan="8">Minimum cost flow problem</th></tr>
<tr><th colspan="5">Transportation method</th><th rowspan="2">Shortest route problem</th><th rowspan="2">Maximum flow problem</th><th rowspan="2">Minimum cost flow problem</th></tr>
<tr><th colspan="3">Transportation method</th><th colspan="2">Assignment method</th></tr>
<tr><td colspan="3">Streamlined simplex method</td><td colspan="2">Hungarian method</td><td rowspan="2">Dijkstra method</td><td rowspan="2">Ford-Fulkerson method</td><td rowspan="2">Networking simplex method</td></tr>
<tr><td>Northwest corner path</td><td>Dantzig method</td><td>Vogel method</td><td>Discrepancy method</td><td>Signing method</td></tr>
</table>

BUDAPEST UNIVERSITY OF TECHNOLOGY AND ECONOMICS

Dr. Tibor SIPOS Ph.D.
Dr. Árpád TÖRÖK Ph.D.
Zsombor SZABÓ

email: szabo.zsombor@mail.bme.hu

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Source of the Images, and the Example

- Hillier F. S, Lieberman G. J. The Maximum Flow Problem. In: Hillier F. S, Lieberman G. J. Introduction to Operations Research. 7th ed. New York: McGrow-Hill; 2001. p. 420-429. ISBN: 0-07-232169-5
- Hillier F. S, Lieberman G. J. The Minimum Cost Flow Problem. In: Hillier F. S, Lieberman G. J. Introduction to Operations Research. 7th ed. New York: McGrow-Hill; 2001. p. 429-438. ISBN: 0-07-232169-5