HUNGARIAN METHOD

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2020

BME FACULTY OF TRANSPORTATION ENGINEERING AND VEHICLE ENGINEERING
32708-2/2017/INTFIN COURSE MATERIAL SUPPORTED BY EMMI

Content

Hungarian Method

↓

Assignment Problem

Hungarian Method

- Degenerated problems cannot be solved easily by Streamlined Simplex Method
- Hungarian Method
- The Hungarian method has got its name from Harold W. Kuhn, who read Dénes König's book where the basic idea - Jenő Egerváry's theorem - was mentioned in a footnote.

Matrix Reduction

- The order:
  - Red
  - Green

	1	2	3	4	5	
1	6 4 3	3 1 0	5 3 3	2 0 0	7 5 5	2
2	3 2 1	7 6 5	4 3 3	4 3 3	1 0 0	1
3	5 4 3	2 1 0	3 2 2	1 0 0	6 5 5	1
4	3 1 0	5 3 2	2 0 0	3 1 1	2 0 0	2
	1	1	0	0	0	

Matrix Reduction

	1	2	3	4	5
1	3	0	3	0	5
2	1	5	3	3	0
3	3	0	2	0	5
4	0	2	0	1	0

Initial Solution

c_{ij}	1	2	3	4	5	
1	3	0	3	0	5	200
2	1	5	3	3	0	80
3	3	0	2	0	5	130
4	0	2	0	1	0	90
	30	210	60	80	120	

Initial Solution

c_{ij}	1	2	3	4	5	
1	3	200 0	3	0	5	0
2	1	5	3	3	0	80
3	3	0	2	0	5	130
4	0	2	0	1	0	90
	30	10	60	80	120	

Initial Solution

c_{ij}	1	2	3	4	5	
1	3	200 0	3	0	5	0
2	1	5	3	3	80 0	0
3	3	0	2	0	5	130
4	0	2	0	1	0	90
	30	10	60	80	40	

Initial Solution

c_{ij}	1	2	3	4	5	
1	3	200 0	3	0	5	0
2	1	5	3	3	80 0	0
3	3	10 0	2	0	5	120
4	0	2	0	1	0	90
	30	0	60	80	40	

Initial Solution

c_{ij}	1	2	3	4	5	
1	3	200 0	3	0	5	0
2	1	5	3	3	80 0	0
3	3	10 0	2	80 0	5	40
4	0	2	0	1	0	90
	30	0	60	0	40	

Initial Solution

c_{ij}	1	2	3	4	5	
1	3	200 0	3	0	5	0
2	1	5	3	3	80 0	0
3	3	10 0	2	80 0	5	40
4	30 0	2	0	1	0	60
	0	0	60	0	40	

Initial Solution

c_{ij}	1	2	3	4	5	
1	3	200 0	3	0	5	0
2	1	5	3	3	80 0	0
3	3	10 0	2	80 0	5	40
4	30 0	2	60 0	1	0	0
	0	0	0	0	40	

Hungarian method – Covering system

- Discrepancy: the remaining possible flow in each row and column, after the programming method
- Goal of the coverage step is to cover all of the zeros, with the minimum possible number of lines
- Cover all the columns, where the discrepancy is zero

Covering system

	1	2	3	4	5	
1	3	0 (200)	3	0	5	0
2	1	5	3	3	0 (80)	0
3	3	0 (10)	2	0 (80)	5	40
4	0 (30)	2	0 (60)	1	0	0
	0	0	0	0	40	

Hungarian method – Covering system

- Whether the uncovered zero's row's discrepancy is zero
- Choose the non-covered zeros
- Cover their rows
- Uncover some columns to avoid double covered zeros
- Star the double covered zeros

- Whether the uncovered zero's row's discrepancy is not zero
- Zeros cannot be covered to fulfill the rules of the covering system
- Repairing path is need to be found

Covering system

	1	2	3	4	5	
1	3	200 0	3	0	5	0
2	1	5	3	3	80 0	0
3	3	10 0	2	80 0	5	40
4	30 0	2	60 0	1	0	0
	0	0	0	0	40	

Covering system

	1	2	3	4	5	
1	3	0 (200)	3	0	5	0
2	1	5	3	3	0 (80)	0
3	3	0 (10)	2	0 (80)	5	40
4	0 (30)	2	0 (60)	1	0	0
	0	0	0	0	40	

Covering system

	1	2	3	4	5	
1	3	200 0	3	0	5	0
2	1	5	3	3	80 0	0
3	3	10 0	2	80 0	5	40
4	* 30 0	2	* 60 0	1	0	0
	0	0	0	0	40	

Hungarian method – discrepancy is 0

- The minimum of the uncovered elements have to be chosen
- Have to be extracted from the uncovered elements
- Have to be added to the twice covered elements

Previous matrix

	1	2	3	4	5
1	3	0	3	0	5
2	1	5	3	3	0
3	3	0	2	0	5
4	0	2	0	1	0

New matrix

	1	2	3	4	5
1	1	0	1	0	3
2	1	7	3	5	0
3	1	0	0	0	3
4	0	4	0	3	0

Initial Solution

	1	2	3	4	5	
1	1	200 0	1	0	3	0
2	1	7	3	5	80 0	0
3	1	10 0	60 0	60 0	3	0
4	30 0	4	0	3	40 0	20
	0	0	0	20	0	

Covering System

	1	2	3	4	5	
1	1	200 0	1	0	3	0
2	1	7	3	5	80 0	0
3	1	10 0	60 0	60 0	3	0
4	30 0	4	0	3	40 0	20
	0	0	0	20	0	

Covering System

	1	2	3	4	5	
1	1	* 200 0	1	0	3	0
2	1	7	3	5	80 0	0
3	1	10 0	60 0	60 0	3	0
4	30 0	4	0	3	40 0	20
	0	0	0	20	0	

Covering System

	1	2	3	4	5	
1	1	* 200 0	1	0	3	0
2	1	7	3	5	80 0	0
3	1	10 0	* 60 0	60 0	3	0
4	30 0	4	0	3	40 0	20
	0	0	0	20	0	

Repairing path

- Z_0: the uncovered zero element
- Z_1: starred zero element in Z_0's column
- Z_2: unstarred zero element in Z_1's row
- Continue this method, until the chosen unstarred zero is in the column, where the discrepancy is larger than 0
- Then choose the minimum of the discrepancies and the starred zeros' flows (now 20)
- All of the starred cells must be decreased by the minimum, while the unstarred zeros must be increased by this value, then all of the coverage lines must be deleted

Repairing path

- Z_0: cell 43
- Z_1: cell 33
- Z_2: cell 32
- Z_3: cell 12
- Z_4: cell 14

	1	2	3	4	5	
1	1	* 200 0	1	0	3	0
2	1	7	3	5	80 0	0
3	1	10 0	* 60 0	60 0	3	0
4	30 0	4	0	3	40 0	20
	0	0	0	20	0	

Optimal solution

	1	2	3	4	5	
1	1	* 180 0	1	20 0	3	0
2	1	7	3	5	80 0	0
3	1	30 0	* 40 0	60 0	3	0
4	30 0	4	20 0	3	40 0	0
	0	0	0	0	0	

Alternate optimum – Repairing path

- Z_0: cell 43
- Z_1: cell 33
- Z_2: cell 34

	1	2	3	4	5	
1	1	* 200 / 0	1	0	3	0
2	1	7	3	5	80 / 0	0
3	1	10 / 0	* 60 / 0	60 / 0	3	0
4	30 / 0	4	0	3	40 / 0	20
	0	0	0	20	0	

Alternate optimum – Optimal solution

	1	2	3	4	5	
1	1	* 200 0	1	0	3	0
2	1	7	3	5	80 0	0
3	1	10 0	* 40 0	80 0	3	0
4	30 0	4	20 0	3	40 0	0
	0	0	0	0	0	

Alternate optimum – Covering System

	1	2	3	4	5	
1	1	0 (200)	1	0	3	0
2	1	7	3	5	0 (80)	0
3	1	0 (10)	0 (60)	0 (60)	3	0
4	0 (30)	4	0	3	0 (40)	20
	0	0	0	20	0	

Alternate optimum – Covering System

	1	2	3	4	5	
1	1	200 0	1	0	3	0
2	1	7	3	5	80 0	0
3	1	* 10 0	* 60 0	60 0	3	0
4	30 0	4	0	3	40 0	20
	0	0	0	20	0	

Alternate optimum – Covering System

	1	2	3	4	5	
1	1	0 (200)	1	0	3	0
2	1	7	3	5	0 (80)	0
3	1	0 (* 10)	0 (* 60)	0 (60)	3	0
4	0 (30)	4	0	3	0 (40)	20
	0	0	0	20	0	

Alternate optimum – Repairing path

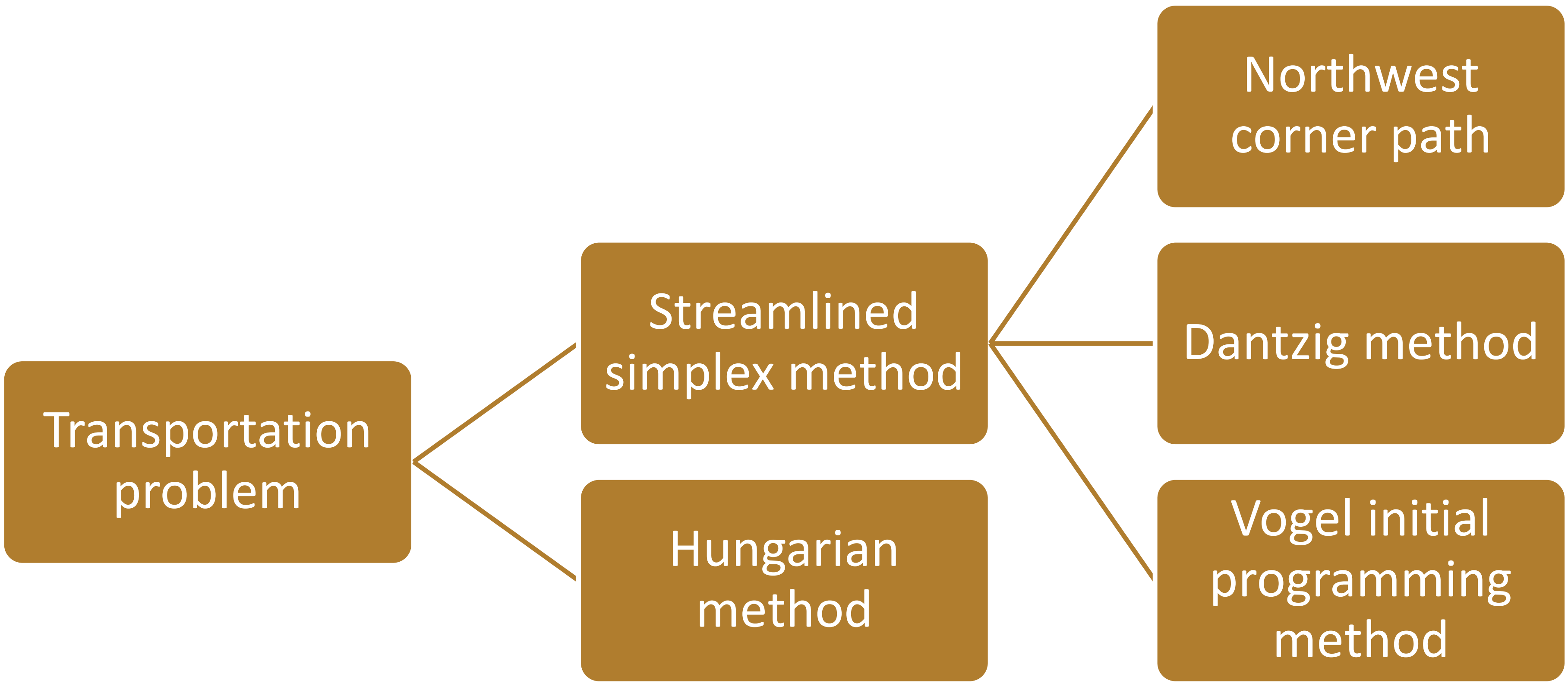
- Z_0: cell 43
- Z_1: cell 33
- Z_2: cell 34

	1	2	3	4	5	
1	1	200 0	1	0	3	0
2	1	7	3	5	80 0	0
3	1	* 10 0	* 60 0	60 0	3	0
4	30 0	4	0	3	40 0	20
	0	0	0	20	0	

Alternate optimum – Optimal solution

	1	2	3	4	5	
1	1	200 0	1	0	3	0
2	1	7	3	5	80 0	0
3	1	* 10 0	* 40 0	80 0	3	0
4	30 0	4	20 0	3	40 0	0
	0	0	0	0	0	

Conclusion


Transportation problem
Streamlined simplex method
Hungarian method
Northwest corner path
Dantzig method
Vogel initial programming method

Content

Hungarian Method

Assignment Problem

Assignment method

- Aim: choose an optimal assignment of n men to n jobs
  - Numerical ratings are given for each man's performance on each job
  - The investigated process should be investigated rather from a performance-like point of view, the objective function might be a maximize function

$$c'_{ij} = \max_{i,j} c_{ij} - c_{ij} \ \forall\, i, j$$

- The assignment problem is a special case of the transportation problem, with two constraints
  - $a_i = b_j = 1 \ \forall\, i, j$
  - $x_{ij} = \begin{cases} 0 \\ 1 \end{cases} \forall\, i, j$

Linear Programming Method

$$\min Z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

$$\sum_{j=1}^{n} x_{ij} = 1 \; \forall i$$

$$\sum_{i=1}^{m} x_{ij} = 1 \; \forall j$$

$$x_{ij} = \begin{cases} 0 \\ 1 \end{cases} \forall \, i, j$$

Example problem

	1	2	3	4
1	6	1	3	4
2	2	5	7	1
3	4	1	2	6
4	5	2	4	8

Feasible Methods

- As the assignment problem is a degenerated transportation problem, the most conveniant way to solve, is the Hungarian method
- Two approaches:
  - Discrepancy method (as a ‚classic Hungarian method’)
  - Signing method (special method for the assignment problems)

Matrix reduction

- Same as the Hungarian method
- The sum of the minimum of each row's (5), and at the second step the sum of the minimum of each column's (2) summary (7) is considered as a lower boundary of the optimal solution
- The order:
  - Red
  - Green

	1	2	3	4	
1	6 5 4	1 0 0	3 2 1	4 3 3	1
2	2 1 0	5 4 4	7 6 5	1 0 0	1
3	4 3 2	1 0 0	2 1 0	6 5 5	1
4	5 3 2	2 0 0	4 2 1	8 6 6	2
	1	0	1	0	5 2

Programming

- Feasible cell: $c_{ij} = 0$
- Tied elements: $x_{ij} = 1$
- If there are not any starred zeros in the column or the row of the chosen feasible cell, then we star it
- Starred zeros (the tied elements)
- Non-starred feasible cells

	1	2	3	4
1	4	0	1	3
2	0	4	5	0
3	2	0	0	5
4	2	0	1	6

Programming

	1	2	3	4
1	4	0	1	3
2	0	4	5	0
3	2	0	0	5
4	2	0	1	6

Programming

	1	2	3	4
1	4	0	1	3
2	0	4	5	0
3	2	0	0	5
4	2	0	1	6

Programming

	1	2	3	4
1	4	0	1	3
2	0	4	5	0
3	2	0	0	5
4	2	0	1	6

Programming

	1	2	3	4
1	4	0	1	3
2	0	4	5	0
3	2	0	0	5
4	2	0	1	6

Programming

	1	2	3	4
1	4	0	1	3
2	0	4	5	0
3	2	0	0	5
4	2	0	1	6

Programming

	1	2	3	4
1	4	0	1	3
2	0	4	5	0
3	2	0	0	5
4	2	0	1	6

Programming

	1	2	3	4
1	4	0	1	3
2	0	4	5	0
3	2	0	0	5
4	2	0	1	6

Coverage

- The coverage step is also the same as in the transportation problem
- Those programmed cells (starred zeros - cells with red background) need to be covered where the discrepancy is zero
- Similarly to the previously presented modell in the repairing section the starred and the unstarred zeros must be changed
- Special method for assignment problem

Labelling method

- Goal: create a covering system
- The first step is to label the row, where there are no tied elements, marked by a black '-' sign
- Find the unstarred zero in the marked row (there must be at least one)
- The column where the unstarred zero is located (in the marked row) must be labelled with the number of the row
- Rows of starred zeros (cells with red background) located in a labelled column must be marked by the number of the column (indicated by green)
- Must be continued until all the rows and columns have marked, or there is no more unstarred zero in the starred zero's row
- Two available results
  - Labelling ends in row
  - Labelling ends in column

Labelling method

	1	2	3	4	sign
1	4	0	1	3	
2	0	4	5	0	
3	2	0	0	5	
4	2	0	1	6	-
sign					

Labelling method

	1	2	3	4	sign
1	4	0	1	3	
2	0	4	5	0	
3	2	0	0	5	
4	2	0	1	6	-
sign		4			

Labelling method

	1	2	3	4	sign
1	4	0	1	3	2
2	0	4	5	0	
3	2	0	0	5	
4	2	0	1	6	-
sign		4			

Labelling ends in row

- Have to cover
  - Signed columns
  - Unsigned rows
- The minimum of the uncovered elements is need to be chosen
  - Extract from the uncovered elements
  - Add to the twice covered elements

	1	2	3	4	sign
1	4	0	1	3	2
2	0	4	5	0	
3	2	0	0	5	
4	2	0	1	6	-
sign		4			

Labelling ends in row

	1	2	3	4
1	3	0	0	2
2	0	5	5	0
3	2	1	0	5
4	1	0	0	5

Latter steps

	1	2	3	4	sign
1	3	0	0	2	
2	0	5	5	0	
3	2	1	0	5	
4	1	0	0	5	
sign					

Latter steps

	1	2	3	4	sign
1	3	0	0	2	2
2	0	5	5	0	
3	2	1	0	5	3
4	1	0	0	5	-
sign		4	4		

Latter steps

	1	2	3	4	sign
1	3	0	0	2	2
2	0	5	5	0	
3	2	1	0	5	3
4	1	0	0	5	-
sign		4	4		

Latter steps

	1	2	3	4
1	2	0	0	1
2	0	6	6	0
3	1	1	0	4
4	0	0	0	4

Latter steps

	1	2	3	4	sign
1	2	0	0	1	
2	0	6	6	0	
3	1	1	0	4	
4	0	0	0	4	
sign					

Latter steps

	1	2	3	4	sign
1	2	0	0	1	2
2	0	6	6	0	1
3	1	1	0	4	3
4	0	0	0	4	-
sign	4	4	4	2	

Labelling ends in column

- Covering system cannot be formed
- Have to define where the labelling process ends
- The label number's row (2nd row) must be chosen
- The starred zero must be modified to unstarred and vice versa
- So the last column's (where the labeling ends) unstarred zero will be starred
- In this case one of the unstarred zeros in the '-' labelled row can be starred
- Technically a reparing path is found

	1	2	3	4	sign
1	2	0	0	1	2
2	0	6	6	0	1
3	1	1	0	4	3
4	0	0	0	4	-
sign	4	4	4	2	

Labelling ends in column

	1	2	3	4
1	2	0	0	1
2	0	6	6	0
3	1	1	0	4
4	0	0	0	4

Solution

	1	2	3	4
1	6	1	3	4
2	2	5	7	1
3	4	1	2	6
4	5	2	4	8

Alternative solution – Discrepancy method

	1	2	3	4
1	4	0	1	3
2	0	4	5	0
3	2	0	0	5
4	2	0	1	6

Alternative solution – Discrepancy method

	1	2	3	4
1	4	0	1	3
2	0	4	5	0
3	2	0	0	5
4	2	0	1	6

Alternative solution – Discrepancy method

	1	2	3	4
1	4	0	1	3
2	0	4	5	0
3	2	0	0	5
4	2	0	1	6

Alternative solution – Discrepancy method

	1	2	3	4
1	4	0	1	3
2	0	4	5	0
3	2	0	0	5
4	2	0	1	6

Reminder

	1	2	3	4
1	4	0	1	3
2	0	4	5	0
3	2	0	0	5
4	2	0	1	6

Discrepancy method

	1	2	3	4
1	4	0	1	3
2	0	4	5	0
3	2	0	0	5
4	2	0	1	6

Labelling method

Alternative solution – Discrepancy method

	1	2	3	4
1	2	0	1	1
2	0	6	7	0
3	0	0	0	3
4	0	0	1	4

Alternative solution – Discrepancy method

	1	2	3	4
1	2	0	1	1
2	0	6	7	0
3	0	0	0	3
4	0	0	1	4

Alternative solution – Discrepancy method

	1	2	3	4
1	2	0	1	1
2	0	6	7	0
3	0	0	0	3
4	0	0	1	4

Alternative solution – Discrepancy method

	1	2	3	4
1	2	0	1	1
2	0	6	7	0
3	0	0	0	3
4	0	0	1	4

Alternative solution – Discrepancy method

	1	2	3	4
1	2	0	1	1
2	0	6	7	0
3	0	0	0	3
4	0	0	1	4

Alternative solution – Discrepancy method

	1	2	3	4
1	2	0	1	1
2	0	6	7	0
3	0	0	0	3
4	0	0	1	4

Alternative solution – Discrepancy method

	1	2	3	4
1	2	0	1	1
2	0	6	7	0
3	0	0	0	3
4	0	0	1	4

Alternative solution – Discrepancy method

	1	2	3	4
1	6	1	3	4
2	2	5	7	1
3	4	1	2	6
4	5	2	4	8

Conclusion

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