DECISION-MAKING METHODS IN TRANSPORTATION
THE SIMPLEX METHOD (II.)
TWO-PHASE SIMPLEX METHOD & DUALITY THEORY

Dr. Tibor SIPOS Ph.D.

Anteneh AFEWORK

Zsombor SZABÓ

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BME FACULTY OF TRANSPORTATION ENGINEERING AND VEHICLE ENGINEERING

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Summary of the previous lecture

General Linear Programming Problems:

Maximize / Minimize:

$$Z = c_1x_1 + c_2x_2 + c_3x_3 + \ldots\ldots + c_nx_n$$

Subject to constraints:

$$a_{11}x_1 + a_{12}x_2 + \ldots\ldots\ldots\ldots + a_{1n}x_n \ (\leq \text{ or } \geq)\ b_1$$

$$a_{21}x_1 + a_{22}x_2 + \ldots\ldots\ldots\ldots + a_{2n}x_n \ (\leq \text{ or } \geq)\ b_2$$

....

$$a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots\ldots + a_{mn}x_n \ (\leq \text{ or } \geq)\ b_m$$

and

$$x_1 \geq 0,\ x_2 \geq 0, \ldots,\ x_n \geq 0$$

conversion (4 steps) →

Standard form (SLPP)

↓ *Simplex method (a…g steps)*

Optimal solution

- *a. Standard form*
- *b. Introducing slack/surplus variables*
- *c. Creating the table (+check optimality)*
- *d. Pivot variables*
- *e. Creating a new table*
- *f. Checking for optimality*
- *g. Identify optimal values*

Introduction to the Two-phase Simplex method

This method is able to find a starting basic feasible solution whenever it exists.

Phases of the Two-phase Simplex method:

1. In the first phase the algorithm tries to determine an initial basic feasible solution. To do this, artificial variables are introduced in phase 1 and dropped, when beginning the second phase.
2. If the constraints are feasible, then the basic feasible solution from the end of phase 1 is used in phase 2 to begin a search for the optimal solution.

Solving a linear programming model using the Two-phase Simplex Method

Minimize $Z = 6x_1 + 3x_2$

Subject to

$$x_1 + x_2 \geq 1$$

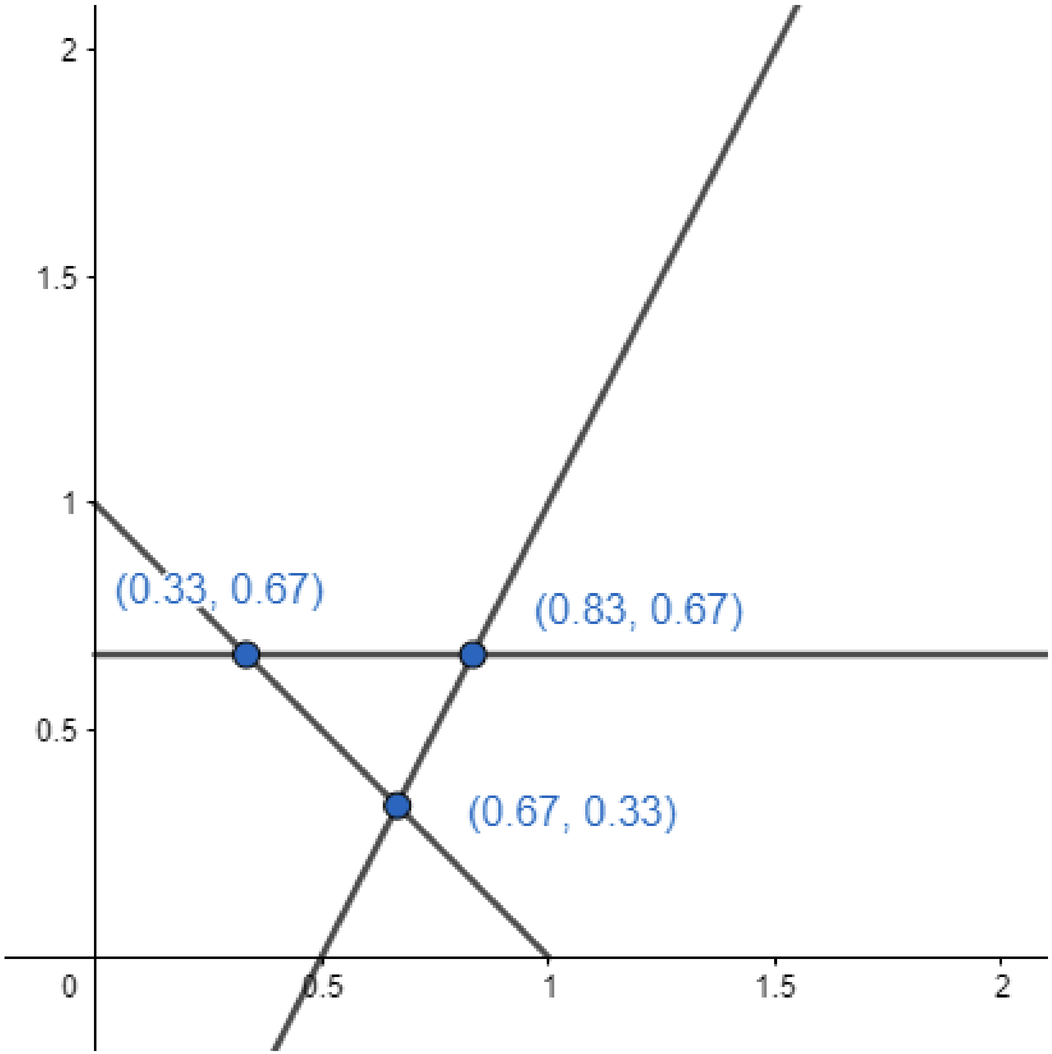
$$2x_1 - x_2 \geq 1$$

$$3x_2 \leq 2$$

$$\text{and } x_1, x_2 \geq 0$$

a. Standard form
b. Introducing slack/surplus variables
Phase 1
c. Introducing new artifical variables and change objective func.
d. Creating table
e. Pivot variable
f. Creating a new table (+checking optimality)
g. Identifying initial BFS
Phase 2
h. Dropping the new artifical variables and change back obj.func.
i. Solve with Simplex

Solving a linear programming model using the Two-phase Simplex Method



Solving a linear programming model using the Two-phase Simplex Method

Step a,b – *Standard form+Introducing slack/surplus variables*

= the 4 steps of converting the GLPP to SLPP!

Minimize $Z = 6x_1 + 3x_2$

Subject to

$x_1 + x_2 \geq 1$

$2x_1 - x_2 \geq 1$

$3x_2 \leq 2$

and $x_1, x_2 \geq 0$

Maximize $(-Z) = -6x_1 - 3x_2$

Subject to

$x_1 + x_2 - s_1 = 1$

$2x_1 - x_2 - s_2 = 1$

$3x_2 + s_3 = 2$

and $x_1, x_2, s_1, s_2, s_3 \geq 0$

a. Standard form

b. Introducing slack/surplus variables

Phase 1

c. Introducing new artifical variables and change objective func.

d. Creating table

e. Pivot variable

f. Creating a new table (+checking optimality)

g. Identifying initial BFS

Phase 2

h. Dropping the new artifical variables and change back obj.func.

i. Solve with Simplex

Solving a linear programming model using the Two-phase Simplex Method (PHASE 1)

Step c – *Introducing new artifical variables and change the objective function*

- Add new artifical variables (y_i) where we had surplus variables
- Change the objective function to minimize the sum of the new artifical variables

Maximize

$(-Z) = -6x_1 - 3x_2$

Subject to

$x_1 + x_2 - s_1 = 1$

$2x_1 - x_2 - s_2 = 1$

$3x_2 + s_3 = 2$

and $x_1, x_2, s_1, s_2, s_3 \geq 0$

→

Minimize

$y_1 + y_2$

Subject to

$x_1 + x_2 - s_1 + y_1 = 1$

$2x_1 - x_2 - s_2 + y_2 = 1$

$3x_2 + s_3 = 2$

$x_1, x_2, s_1, s_2, s_3, y_1\ y_2 \geq 0$

a. Standard form
b. Introducing slack/surplus variables
Phase 1
c. Introducing new artifical variables and change objective func.
d. Creating table
e. Pivot variable
f. Creating a new table (+checking optimality)
g. Identifying initial BFS
Phase 2
h. Dropping the new artifical variables and change back obj.func.
i. Solve with Simplex

Solving a linear programming model using the Two-phase Simplex Method (PHASE 1)

Step d – *Creating the table (initial table of Phase 1)*

- coefficients corresponding to the linear constraint variables
- coefficients of the objective functions (same as in case of Simplex related to Phase 2; write the objective in terms of non-basic variables related to Phase 1: (-1) in the columns of y_i, converted to 0 by row operations)

Phase 1
c. Introducing new artifical variables and change objective func.
d. Creating table
e. Pivot variable
f. Creating a new table (+checking optimality)
g. Identifying initial BFS

Original objective function:

Maximize $(-Z) = -6x_1 - 3x_2$

Objective function of Phase 1:

Minimize y_1+y_2

Subject to

$x_1 + x_2 - s_1 + y_1 = 1$

$2x_1 - x_2 - s_2 + y_2= 1$

$3x_2 + s_3 = 2$

$x_1, x_2, s_1, s_2, s_3, y_1\ y_2 \geq 0$

	x_1	x_2	s_1	s_2	s_3	y_1	y_2	b
	1	1	-1	0	0	1	0	1
	2	-1	0	-1	0	0	1	1
	0	3	0	0	1	0	0	2
Ph. II.	6	3	0	0	0	0	0	0
Ph. I.	3	0	-1	-1	0	0	0	2

Solving a linear programming model using the Two-phase Simplex Method (PHASE 1)

Step e – *Identifying the pivot variable*

- **Identifying pivot variable using the table**: in the column of the highest positive value in bottom row (because we are minimizing now, not maximizing); in the row of the smallest non-negative indicator (indicator: divide the beta values of the linear constraints by their corresponding values from the column containing the possible pivot variable)

Phase 1
- *c. Introducing new artifical variables and change objective func.*
- *d. Creating table*
- ***e. Pivot variable***
- *f. Creating a new table (+checking optimality)*
- *g. Identifying initial BFS*

	x_1	x_2	s_1	s_2	s_3	y_1	y_2	b	Indicator
	1	1	-1	0	0	1	0	1	1/1
	2	-1	0	-1	0	0	1	1	1/2
	0	3	0	0	1	0	0	2	-
Ph. II.	6	3	0	0	0	0	0	0	
Ph. I.	3	0	-1	-1	0	0	0	2	

Highest pos. value

Smallest non-negative indicator

Solving a linear programming model using the Two-phase Simplex Method (PHASE 1)

Step f– *Creating the new table*

I. To optimize the pivot variable, it will need to be transformed into a unit value (value of 1)

II. The other values in the column containing the unit value have to become zero

III. During this, the table have to be kept equivalent

Phase 1

c. Introducing new artifical variables and change objective func.

d. Creating table

e. Pivot variable

f. Creating a new table
(+checking optimality)

g. Identifying initial BFS

New tableau value = (Negative value in old tableau pivot column) x (value in new tableau pivot row) + (Old tableau value)

➢ *The new table will be used to identify a new possible optimal solution*

Solving a linear programming model using the Two-phase Simplex Method (PHASE 1)

Step f – *Creating the new table*

I. To optimize the pivot variable, it will need to be transformed into a unit value (value of 1)
II. The other values in the column containing the unit value have to become zero
III. During this, the table have to be kept equivalent

New tableau value = (Negative value in old tableau pivot column) x (value in new tableau pivot row) + (Old tableau value) (for example: 3 = (-6)*(-1/2)+0)

The initial table:

	x_1	x_2	s_1	s_2	s_3	y_1	y_2	b
	1	1	-1	0	0	1	0	1
	2	-1	0	-1	0	0	1	1
	0	3	0	0	1	0	0	2
Ph. II.	6	3	0	0	0	0	0	0
Ph. I.	3	0	-1	-1	0	0	0	2

Old pivot column

The new table:

	x_1	x_2	s_1	s_2	s_3	y_1	y_2	b
	0	3/2	-1	1/2	0	1	-1/2	1/2
	1	-1/2	0	-1/2	0	0	1/2	1/2
	0	3	0	0	1	0	0	2
Ph. II.	0	6	0	3	0	0	-3	-3
Ph. I.	0	3/2	-1	1/2	0	0	-3/2	1/2

New pivot row

Solving a linear programming model using the Two-phase Simplex Method (PHASE 1)

Step f – *(+checking for optimality)*

- **Check optimality using the table**: all values in the last row must contain values less than or equal to zero (because we are minimizing in Phase 1, not maximizing)

Phase 1

c. Introducing new artifical variables and change objective func.

d. Creating table

e. Pivot variable

f. Creating a new table ***(+checking optimality)***

g. Identifying initial BFS

	x_1	x_2	s_1	s_2	s_3	y_1	y_2	b
	0	3/2	-1	1/2	0	1	-1/2	1/2
	1	-1/2	0	-1/2	0	0	1/2	1/2
	0	3	0	0	1	0	0	2
Ph. II.	0	6	0	3	0	0	-3	-3
Ph. I.	0	3/2	-1	1/2	0	0	-3/2	1/2

Not optimal yet

Solving a linear programming model using the Two-phase Simplex Method (PHASE 1)

Step e (again) – *Identifying the pivot variable*

- **Identifying pivot variable using the table**: in the column of the highest positive value in bottom row (because we are minimizing now, not maximizing); in the row of the smallest non-negative indicator (indicator: divide the beta values of the linear constraints by their corresponding values from the column containing the possible pivot variable)

Phase 1

c. Introducing new artifical variables and change objective func.

d. Creating table

e. Pivot variable

f. Creating a new table (+checking optimality)

g. Identifying initial BFS

	x_1	x_2	s_1	s_2	s_3	y_1	y_2	b	Indicator
	0	**3/2**	-1	1/2	0	1	-1/2	1/2	1/3
	1	-1/2	0	-1/2	0	0	1/2	1/2	-1
	0	3	0	0	1	0	0	2	2/3
Ph. II.	0	6	0	3	0	0	-3	-3	
Ph. I.	0	3/2	-1	1/2	0	0	-3/2	1/2	

Highest pos. value

Smallest non-negative indicator

Solving a linear programming model using the Two-phase Simplex Method (PHASE 1)

Step f (again) – *Creating the new table (+checking optimality)*

I. To optimize the pivot variable, it will need to be transformed into a unit value (value of 1)
II. The other values in the column containing the unit value have to become zero
III. During this, the table have to be kept equivalent

New tableau value = (Negative value in old tableau pivot column) x (value in new tableau pivot row) + (Old tableau value) (for example: 2/3 = (+1/2)*(1/3)+(1/2))

The old table:

	x_1	x_2	s_1	s_2	s_3	y_1	y_2	b
	0	3/2	-1	1/2	0	1	-1/2	1/2
	1	-1/2	0	-1/2	0	0	1/2	1/2
	0	3	0	0	1	0	0	2
Ph. II.	0	6	0	3	0	0	-3	-3
Ph. I.	0	3/2	-1	1/2	0	0	-3/2	1/2

Old pivot column

The new table:

	x_1	x_2	s_1	s_2	s_3	y_1	y_2	b
	0	1	-2/3	1/3	0	2/3	-1/3	1/3
	1	0	-1/3	-1/3	0	1/3	1/3	2/3
	0	0	2	-1	1	-2	1	1
Ph. II.	0	0	4	1	0	-4	-1	-5
Ph. I.	0	0	0	0	0	-1	-1	0

New pivot row

Optimality reached

Solving a linear programming model using the Two-phase Simplex Method (PHASE 1)

Step g– *Identifying initial BFS*

- **Basic variable**: have a single 1 value in its column and the rest are zeros; the row that contains the 1 value will correspond to the beta value. The beta value will represent the optimal solution for the given variable
- **Non-basic variable**: the remaining variables; the optimal solution of the non-basic variables is zero

Phase 1

c. Introducing new artifical variables and change objective func.

d. Creating table

e. Pivot variable

f. Creating a new table (+checking optimality)

g. Identifying initial BFS

	x_1	x_2	s_1	s_2	s_3	y_1	y_2	b
	0	1	-2/3	1/3	0	2/3	-1/3	1/3
	1	0	-1/3	-1/3	0	1/3	1/3	2/3
	0	0	2	-1	1	-2	1	1
Ph. II.	0	0	4	1	0	-4	-1	-5
Ph. I.	0	0	0	0	0	-1	-1	0

Initial BFS:

$x_1 = 2/3$ $y_1 = 0$

$x_2 = 1/3$ $y_2 = 0$

$s_1 = 0$

$s_2 = 0$

$s_3 = 1$

Step h – *Dropping the new artifical values and change back objective function*

- Drop the columns of the new artificial variables and the row corresponding to the objective of the first phase

	x_1	x_2	s_1	s_2	s_3	y_1	y_2	b
	0	1	-2/3	1/3	0	2/3	-1/3	1/3
	1	0	-1/3	-1/3	0	1/3	1/3	2/3
	0	0	2	-1	1	-2	1	1
Ph. II.	0	0	4	1	0	-4	-1	-5
Ph. I.	0	0	0	0	0	-1	-1	0

↓

	x_1	x_2	s_1	s_2	s_3	b
	0	1	-2/3	1/3	0	1/3
	1	0	-1/3	-1/3	0	2/3
	0	0	2	-1	1	1
Ph. II.	0	0	4	1	0	-5

a. Standard form
b. Introducing slack/surplus variables
Phase 1
c. Introducing new artifical variables and change objective func.
d. Creating table
e. Pivot variable
f. Creating a new table (+checking optimality)
g. Identifying initial BFS
Phase 2
h. Dropping the new artifical variables and change back obj.func.
i. Solve with Simplex

Solving a linear programming model using the Two-phase Simplex Method (PHASE 2)

Step h – *Solve with Simplex Method*

- As we included the original objective as an added row in the tables of phase 1, it is in the right form
- Since we have the initial table, we check for optimality (are there any negative values in the bottom row related to the variables?)
  - If no: we have reached optimality, identify optimal values
  - If yes: Choose pivot, create new table, check optimality again, and so on.

	x_1	x_2	s_1	s_2	s_3	b
	0	1	-2/3	1/3	0	1/3
	1	0	-1/3	-1/3	0	2/3
	0	0	2	-1	1	1
Ph. II.	0	0	4	1	0	-5

Optimality reached

a. Standard form
b. Introducing slack/surplus variables
Phase 1
c. Introducing new artifical variables and change objective func.
d. Creating table
e. Pivot variable
f. Creating a new table (+checking optimality)
g. Identifying initial BFS
Phase 2
h. Dropping the new artifical variables and change back obj.func.
i. Solve with Simplex

Solving a linear programming model using the Two-phase Simplex Method (PHASE 2)

Final solution of the example:

	x_1	x_2	s_1	s_2	s_3	b
	0	1	-2/3	1/3	0	1/3
	1	0	-1/3	-1/3	0	2/3
	0	0	2	-1	1	1
Ph. II.	0	0	4	1	0	-5

Optimality reached

Original task:

Maximize $(-Z) = -6x_1 - 3x_2$

Subject to

$x_1 + x_2 - s_1 = 1$

$2x_1 - x_2 - s_2 = 1$

$3x_2 + s_3 = 2$

$x_1, x_2, s_1\ s_2, s_3 \geq 0$

Solution:

Max(-Z) = -5 → Min Z = 5

Subject to

$x_1 = 2/3$

$x_2 = 1/3$

$s_3 = 1$

$s_1 = s_2 = 0$

a. Standard form
b. Introducing slack/surplus variables
Phase 1
c. Introducing new artifical variables and change objective func.
d. Creating table
e. Pivot variable
f. Creating a new table (+checking optimality)
g. Identifying initial BFS
Phase 2
h. Dropping the new artifical variables and change back obj.func.
i. Solve with Simplex

Example 2. (Two-phase Simplex Method)

Solve the following LPP:

Maximize $Z = 2x_1 + 3x_2 + 4x_3$

Subject to

$$3x_1 + 2x_2 + x_3 \leq 10$$

$$2x_1 + 3x_2 + 3x_3 \leq 15$$

$$x_1 + x_2 - x_3 \geq 4$$

and $x_1, x_2, x_3 \geq 0$

a. Standard form
b. Introducing slack/surplus variables

Phase 1

c. Introducing new artifical variables and change objective func.
d. Creating table
e. Pivot variable
f. Creating a new table (+checking optimality)
g. Identifying initial BFS

Phase 2

h. Dropping the new artifical variables and change back obj.func.
i. Solve with Simplex

Example 2. (Two-phase Simplex Method- Phase 1)

Step a,b – *Standard form+Introducing slack/surplus variables*

> ***a. Standard form***
> ***b. Introducing slack/surplus variables***

Maximize $Z = 2x_1 + 3x_2 + 4x_3$

Subject to

$3x_1 + 2x_2 + x_3 \leq 10$

$2x_1 + 3x_2 + 3x_3 \leq 15$

$x_1 + x_2 - x_3 \geq 4$

and $x_1, x_2, x_3 \geq 0$

Maximize $Z = 2x_1 + 3x_2 + 4x_3$

Subject to

$3x_1 + 2x_2 + x_3 + s_1 = 10$

$2x_1 + 3x_2 + 3x_3 + s_2 = 15$

$x_1 + x_2 - x_3 - s_3 = 4$

and $x_1, x_2, x_3, s_1, s_2, s_3 \geq 0$

Example 2. (Two-phase Simplex Method- Phase 1)

Step c – *Introducing new artifical variables and change the objective function*

> *a. Standard form*
> *b. Introducing slack/surplus variables*
> *Phase 1*
> ***c. Introducing new artifical variables and change objective func.***
> *d. Creating table*
> *e. Pivot variable*
> *f. Creating a new table (+checking optimality)*
> *g. Identifying initial BFS*

Maximize $Z = 2x_1 + 3x_2 + 4x_3$

Subject to

$3x_1 + 2x_2 + x_3 + s_1 = 10$

$2x_1 + 3x_2 + 3x_3 + s_2 = 15$

$x_1 + x_2 - x_3 - s_3 = 4$

and $x_1, x_2, x_3, s_1\ s_2, s_3 \geq 0$

Minimize y_1

$3x_1 + 2x_2 + x_3 + s_1 = 10$

$2x_1 + 3x_2 + 3x_3 + s_2 = 15$

$x_1 + x_2 - x_3 - s_3 + y_1 = 4$

and $x_1, x_2, x_3, s_1, s_2, s_3 \geq 0$

Example 2. (Two-phase Simplex Method- Phase 1)

Step d – *Creating the table (initial table of Phase 1)*

- coefficients corresponding to the linear constraint variables
- coefficients of the objective functions (same as in case of Simplex related to Phase 2; write the objective in terms of non-basic variables related to Phase 1: (-1) in the columns of y_i, converted to 0 by row operations)

Phase 1

c. Introducing new artifical variables and change objective func.

d. Creating table

e. Pivot variable

f. Creating a new table (+checking optimality)

g. Identifying initial BFS

Original objective function:

Maximize $Z = 2x_1 + 3x_2 + 4x_3$

Objective function of Phase 1:

Minimize y_1

Subject to

$3x_1 + 2x_2 + x_3 + s_1 = 10$

$2x_1 + 3x_2 + 3x_3 + s_2 = 15$

$x_1 + x_2 - x_3 - s_3 + y_1 = 4$

$x_1, x_2, x_3, s_1, s_2, s_3 \geq 0$

	x_1	x_2	x_3	s_1	s_2	s_3	y_1	b
	3	2	1	1	0	0	0	10
	2	3	3	0	1	0	0	15
	1	1	-1	0	0	-1	1	4
Ph. II.	-2	-3	-4	0	0	0	0	0
Ph. I.	1	1	-1	0	0	-1	0	4

Example 2. (Two-phase Simplex Method- Phase 1)

Step e – *Identifying the pivot variable*

- **Identifying pivot variable using the table**: in the column of the highest positive value in bottom row (because we are minimizing now, not maximizing); in the row of the smallest non-negative indicator (indicator: divide the beta values of the linear constraints by their corresponding values from the column containing the possible pivot variable)

Phase 1
- *c. Introducing new artifical variables and change objective func.*
- *d. Creating table*
- ***e. Pivot variable***
- *f. Creating a new table (+checking optimality)*
- *g. Identifying initial BFS*

	x_1	x_2	x_3	s_1	s_2	s_3	y_1	b	Indicator
	(3)	2	1	1	0	0	0	10	10/3
	2	3	3	0	1	0	0	15	15/2
	1	1	-1	0	0	-1	1	4	4/1
Ph. II.	-2	-3	-4	0	0	0	0	0	
Ph. I.	1	1	-1	0	0	-1	0	4	

Smallest non-negative indicator

Highest pos. value (optional if x_1 or x_2)

Example 2. (Two-phase Simplex Method- Phase 1)

Step f – *Creating the new table (+check optimality)*

I. To optimize the pivot variable, it will need to be transformed into a unit value (value of 1)
II. The other values in the column containing the unit value have to become zero
III. During this, the table have to be kept equivalent

The initial table:

	x_1	x_2	x_3	s_1	s_2	s_3	y_1	b
	3	2	1	1	0	0	0	10
	2	3	3	0	1	0	0	15
	1	1	-1	0	0	-1	1	4
Ph. II.	-2	-3	-4	0	0	0	0	0
Ph. I.	1	1	-1	0	0	-1	0	4

Old pivot column

The new table:

	x_1	x_2	x_3	s_1	s_2	s_3	y_1	b
	1	2/3	1/3	1/3	0	0	0	10/3
	0	5/3	7/3	-2/3	1	0	0	25/3
	0	1/3	-4/3	-1/3	0	-1	1	2/3
Ph. II.	0	-5/3	-10/3	2/3	0	0	0	20/3
Ph. I.	0	1/3	-4/3	-1/3	0	-1	0	2/3

New pivot row

Not optimal yet

Example 2. (Two-phase Simplex Method- Phase 1)

Step e (again) – *Identifying the pivot variable*

- **Identifying pivot variable using the table**: in the column of the highest positive value in bottom row (because we are minimizing now, not maximizing); in the row of the smallest non-negative indicator (indicator: divide the beta values of the linear constraints by their corresponding values from the column containing the possible pivot variable)

Phase 1

c. Introducing new artifical variables and change objective func.
d. Creating table
e. Pivot variable
f. Creating a new table (+checking optimality)
g. Identifying initial BFS

	x_1	x_2	s_1	s_2	s_3	y_1	y_2	b	Indicator
	1	2/3	1/3	1/3	0	0	0	10/3	5
	0	5/3	7/3	-2/3	1	0	0	25/3	5
	0	1/3	-4/3	-1/3	0	-1	1	2/3	2
Ph. II.	0	-5/3	-10/3	2/3	0	0	0	20/3	
Ph. I.	0	1/3	-4/3	-1/3	0	-1	0	2/3	

Highest pos. value

Smallest non-negative indicator

Example 2. (Two-phase Simplex Method- Phase 1)

Step f – *Creating the new table (+check optimality)*

I. To optimize the pivot variable, it will need to be transformed into a unit value (value of 1)
II. The other values in the column containing the unit value have to become zero
III. During this, the table have to be kept equivalent

The initial table:

	x_1	x_2	x_3	s_1	s_2	s_3	y_1	b
	1	2/3	1/3	1/3	0	0	0	10/3
	0	5/3	7/3	-2/3	1	0	0	25/3
	0	1/3	-4/3	-1/3	0	-1	1	2/3
Ph. II.	0	-5/3	-10/3	2/3	0	0	0	20/3
Ph. I.	0	1/3	-4/3	-1/3	0	-1	0	2/3

Old pivot column

The new table:

	x_1	x_2	x_3	s_1	s_2	s_3	y_1	b
	1	0	3	1	0	2	-2	2
	0	0	9	1	1	5	-5	5
	0	1	-4	-1	0	-3	3	2
Ph. II.	0	0	-10	-1	0	-5	5	10
Ph. I.	0	0	0	0	0	0	-1	0

New pivot row

Optimality reached

Example 2. (Two-phase Simplex Method- Phase 1)

Step g– *Identifying initial BFS*

- **Basic variable**: have a single 1 value in its column and the rest are zeros; the row that contains the 1 value will correspond to the beta value. The beta value will represent the optimal solution for the given variable
- **Non-basic variable**: the remaining variables; the optimal solution of the non-basic variables is zero

Phase 1

c. Introducing new artifical variables and change objective func.
d. Creating table
e. Pivot variable
f. Creating a new table (+checking optimality)
g. Identifying initial BFS

	x_1	x_2	x_3	s_1	s_2	s_3	y_1	b
	1	0	3	1	0	2	-2	2
	0	0	9	1	1	5	-5	5
	0	1	-4	-1	0	-3	3	2
Ph. II.	0	0	-10	-1	0	-5	5	10
Ph. I.	0	0	0	0	0	0	-1	0

Initial BFS:

$x_1 = 2$

$x_2 = 2$

$x_3 = 0$

$s_1 = 0$

$s_2 = 5$

$s_3 = 0$

$y_1 = 0$

Example 2. (Two-phase Simplex Method- Phase 2)

Step h – *Dropping the new artifical values and change back objective function*

- Drop the columns of the new artificial variables and the row corresponding to the objective of the first phase

	x_1	x_2	x_3	s_1	s_2	s_3	y_1	b
	1	0	3	1	0	2	-2	2
	0	0	9	1	1	5	-5	5
	0	1	-4	-1	0	-3	3	2
Ph. II.	0	0	-10	-1	0	-5	5	10
Ph. I.	0	0	0	0	0	0	1	0

	x_1	x_2	x_3	s_1	s_2	s_3	b
	1	0	3	1	0	2	2
	0	0	9	1	1	5	5
	0	1	-4	-1	0	-3	2
Ph. II.	0	0	-10	-1	0	-5	10

a. Standard form
b. Introducing slack/surplus variables
Phase 1
c. Introducing new artifical variables and change objective func.
d. Creating table
e. Pivot variable
f. Creating a new table (+checking optimality)
g. Identifying initial BFS
Phase 2
h. Dropping the new artifical variables and change back obj.func.
i. Solve with Simplex

Example 2. (Two-phase Simplex Method- Phase 2)

Step h – *Solve with Simplex Method*

- As we included the original objective as an added row in the tables of phase 1, it is in the right form
- Since we have the initial table, we check for optimality (are there any negative values in the bottom row related to the variables?)
  - If no: we have reached optimality, identify optimal values
  - If yes: Choose pivot, create new table, check optimality again, and so on.

	x_1	x_2	x_3	s_1	s_2	s_3	b
	1	0	3	1	0	2	2
	0	0	**9**	1	1	5	5
	0	1	-4	-1	0	-3	2
Ph. II.	0	0	-10	-1	0	-5	10

Not optimal yet

a. Standard form
b. Introducing slack/surplus variables
Phase 1
c. Introducing new artifical variables and change objective func.
d. Creating table
e. Pivot variable
f. Creating a new table (+checking optimality)
g. Identifying initial BFS
Phase 2
h. Dropping the new artifical variables and change back obj.func.
i. Solve with Simplex

Example 2. (Two-phase Simplex Method- Phase 2)

Step h – *Solve with Simplex Method*

- Pivot variable: 9 (in column of x_3, second row)

The old table

	x_1	x_2	x_3	s_1	s_2	s_3	b
	1	0	3	1	0	2	2
	0	0	(9)	1	1	5	5
	0	1	-4	-1	0	-3	2
Ph. II.	0	0	-10	-1	0	-5	10

The new table

	x_1	x_2	x_3	s_1	s_2	s_3	b
	1	0	0	2/3	-1/3	1/3	1/3
	0	0	1	1/9	1/9	5/9	5/9
	0	1	0	-5/9	4/9	-7/9	38/9
Ph. II.	0	0	0	1/9	10/9	5/9	140/9

Optimality reached

a. Standard form
b. Introducing slack/surplus variables
Phase 1
c. Introducing new artifical variables and change objective func.
d. Creating table
e. Pivot variable
f. Creating a new table (+checking optimality)
g. Identifying initial BFS
Phase 2
h. Dropping the new artifical variables and change back obj.func.
i. Solve with Simplex

Example 2. (Two-phase Simplex Method- Phase 2)

Final solution of the example:

	x_1	x_2	x_3	s_1	s_2	s_3	b
	1	0	0	2/3	-1/3	1/3	1/3
	0	0	1	1/9	1/9	5/9	5/9
	0	1	0	-5/9	4/9	-7/9	38/9
Ph. II.	0	0	0	1/9	10/9	5/9	140/9

Original task:

Maximize $Z = 2x_1 + 3x_2 + 4x_3$

Subject to

$3x_1 + 2x_2 + x_3 + s_1 = 10$

$2x_1 + 3x_2 + 3x_3 + s_2 = 15$

$x_1 + x_2 - x_3 - s_3 = 4$

and $x_1, x_2, x_3, s_1\ s_2, s_3 \geq 0$

Solution:

Max Z = 140/9

Subject to

$x_1 = 1/3$

$x_2 = 38/9$

$x_3 = 5/9$

$s_1=s_2=s_3 = 0$

Sensitivity of the solutions and the shadow prices

Let's investigate the following linear problem:

Maximize $Z = x_1 + x_2$

Subject to

$x_1 + 2x_2 \leq 6$

$x_1 - x_2 \leq 3$

and $x_1, x_2 \geq 0$

Standard form

Maximize $Z = x_1 + x_2$

Subject to

$x_1 + 2x_2 + s_1 = 6$

$x_1 - x_2 + s_2 = 3$

and $x_1, x_2, s_1, s_2 \geq 0$

Initial Simplex table:

x_1	x_2	s_1	s_2	Z	b
1	2	1	0	0	6
1	-1	0	1	0	3
-1	-1	0	0	1	0

Not optimal yet

Sensitivity of the solutions and the shadow prices

Table 2:

x_1	x_2	s_1	s_2	Z	b
0	3	1	-1	0	3
1	-1	0	1	0	3
0	-2	0	1	1	3

Not optimal yet

Table 3:

x_1	x_2	s_1	s_2	Z	b
0	1	1/3	-1/3	0	1
1	0	1/3	2/3	0	4
0	0	2/3	1/3	1	5

Optimality reached

Solution: *Max(Z) = 5; Subject to:* $x_1 = 4$; $x_2 = 1$; $s_1 = 0$; $s_2 = 0$

Sensitivity of the solutions and the shadow prices

Lets investigate the following linear problem:

Maximize $Z = x_1 + x_2$

Subject to

$x_1 + 2x_2 \leq 6+\pi_1$

$x_1 - x_2 \leq 3+\pi_2$

and $x_1, x_2 \geq 0$

Standard form

Maximize $Z = x_1 + x_2$

Subject to

$x_1 + 2x_2 + s_1 = 6+\pi_1$

$x_1 - x_2 + s_2 = 3+\pi_2$

and $x_1, x_2, s_1, s_2 \geq 0$

Initial Simplex table:

x_1	x_2	s_1	s_2	Z	b
1	2	1	0	0	$6+\pi_1$
1	-1	0	1	0	$3+\pi_2$
-1	-1	0	0	1	0

Not optimal yet

Sensitivity of the solutions and the shadow prices

Table 2:

x_1	x_2	s_1	s_2	Z	b
0	3	1	-1	0	$3+\pi_1-\pi_2$
1	-1	0	1	0	$3+\pi_2$
0	-2	0	1	1	$3+\pi_2$

Not optimal yet

Table 3:

x_1	x_2	s_1	s_2	Z	b
0	1	1/3	-1/3	0	$1+\pi_1/3-\pi_2/3$
1	0	1/3	2/3	0	$4+\pi_1/3+2\pi_2/3$
0	0	2/3	1/3	1	$5+2\pi_1/3+\pi_2/3$

Optimality reached

Shadow prices

Solution: *Max(Z)* $= 5 + 2\pi_1/3 + \pi_2/3;$

Subject to: $x_1 = 4 + \pi_1/3 + 2\pi_2/3;\ x_2 = 1 + \pi_1/3 - \pi_2/3;\ s_1 = 0;\ s_2 = 0$

Introduction to Duality Theory

- In case of every linear program, we can find another linear program strongly related to it
- These two linear programs should be handled as a pair
- In this case we should call the first linear program the *primal program* and the second linear program the *dual program*
- Importance of the duality
  - It can be decided whether a feasible solution is optimal
  - In some cases the dual problem is easier to solve
  - The solution of the dual problem has some economic meaning

Primal and Dual Programs

Connection between minimization and maximization linear programs:

Primal program:

min $\underline{c}*\underline{x}$

subject to

$\underline{\underline{A}}*\underline{x} \geq b$

$\underline{x} \geq 0$

Dual program:

max $\underline{y}*\underline{b}$

subject to

$\underline{y}*\underline{\underline{A}} \leq c$

$\underline{y} \geq 0$

In case of a primal program and its' dual:

min $\underline{c}*\underline{x}$= max $\underline{y}*\underline{b}$

where $\underline{x}$ is a column vector with *n* elements (primary variables), $\underline{y}$ is a row vector with *m* elements (dual variables; optimal dual variables=shadow prices), $\underline{\underline{A}}$ is an *m x n* matrix, $\underline{c}$ is a row vector with *n* elements, and $\underline{b}$ is a column vector with *m* elements.

Duality Theorem

- The following are the only possible relationships between the primal and dual problems
  - If one problem has feasible solutions and a bounded objective function (and so has an optimal solution), then so does the other problem, so both the weak and strong duality properties are applicable
  - If one problem has feasible solutions and an unbounded objective function (and so no optimal solution), then the other problem has no feasible solutions
  - If one problem has no feasible solutions, then the other problem has either no feasible solutions or an unbounded objective function

Primal and Dual Programs and the shadow prices

Shadow price (dual price): The optimal value of the dual variables. It gives the improvement in the objective function if the constraint is relaxed by one unit. A shadow price is reported for each constraint.

- In the case of a less-than-or-equal constraint, such as a resource constraint, the shadow price gives the value of having one more unit of the resource represented by that constraint.
- In the case of a greater-than-or-equal constraint, such as a minimum production level constraint, the shadow price gives the cost of meeting the last unit of the minimum production target.
- The units (dimensions) of the shadow prices are the units of the objective function divided by the units of the constraint. Knowing the units of the dual prices can be useful when you are trying to interpret what the dual prices mean.

Primal and Dual Programs, practical example

Lets investigate a production process:

We produce breads (x_1), and cakes (x_2). Unit profit of a bread is 5 Ft, unit profit of a cake is 6 Ft. Resources of the production process are flour, water and salt. The dimensions of the resources in the primal program are dkg/piece in case of the flour (a_{11}, a_{12}), l/piece in case of the water (a_{21}, a_{22}) and g/piece in case of the salt (a_{31}, a_{32}). Resources are limited, we have 200 dkg flour, 100 l water and 500 g salt. We need 2 dkg flour, 1 l water and 3 g salt to produce a bread, while 2 dkg flour, 2 l water and 1 g salt to produce a cake.

The aim of us is to determine, how much breads and cakes we have to product from our resources, to maximize our total profit (π) based on the unit profit (p_1,p_2) of a cake and a bread considering the available resources (c).

Primal and Dual Programs, practical example

The introduced example can be written as the following linear problem:

Maximize $\pi = p_1x_1 + p_2x_2$

Subject to

$x_1 * a_{11} + x_2 * a_{12} \leq c_1$

$x_1 * a_{21} + x_2 * a_{22} \leq c_2$

$x_1 * a_{31} + x_2 * a_{32} \leq c_3$

and $x_1, x_2 \geq 0$

substituting the values

Maximize $\pi = 5x_1 + 6x_2$

Subject to

$2x_1 + 2x_2 \leq 200$

$x_1 + 2x_2 \leq 100$

$3x_1 + x_2 \leq 500$

and $x_1, x_2 \geq 0$

If we solve this problem, we get the following solution:

$x_1 = 100$ piece; $x_2 = 0$ piece; $\pi = 500$ Ft.

***Now, lets determine the dual program of this example*.**

Primal and Dual Programs, practical example

Generally:

Primal program:

min $\underline{cx}$

subject to

$\underline{\underline{A}}\underline{x} \geq b$

$\underline{x} \geq 0$

Dual program:

max $\underline{yb}$

subject to

$\underline{y}\underline{\underline{A}} \leq c$

$\underline{y} \geq 0$

The introduced example:

Maximize $p_1x_1 + p_2x_2$

Subject to

$x_1*a_{11}+x_2*a_{12} \leq c_1$

$x_1*a_{21}+x_2*a_{22} \leq c_2$

$x_1*a_{31}+x_2*a_{32} \leq c_3$

and $x_1, x_2 \geq 0$

The corresponding dual program:

Minimize $c_1y_1 + c_2y_2 + c_3y_3$

Subject to

$y_1*a_{11}+y_2*a_{21}+y_3*a_{31} \geq p_1$

$y_1*a_{12}+y_2*a_{22}+y_3*a_{32} \geq p_2$

and $y_1, y_2, y_3 \geq 0$

Primal and Dual Programs, practical example

Units (dimensions):

In case of $\underline{\underline{A}}$: $[a_{11},a_{12}]$= dkg/piece; $[a_{21},a_{22}]$= l/piece; $[a_{31},a_{32}]$= g/piece

In case of $\underline{p}$: $[p_1,p_2]$= Ft/piece

In case of $\underline{c}$: $[c_1]$= dkg; $[c_2]$= l; $[c_3]$= g

In case of $\underline{x}$: $[x_1,x_2]$= piece

[Objective function in primal program]=

=(Ft/piece) * (piece)+ (Ft/piece) * (piece) = Ft

The corresponding dual program:

Minimize $c_1y_1 + c_2y_2 + c_3y_3$

Subject to

$y_1*a_{11}+y_2*a_{21}+y_3*a_{31} \geq p_1$

$y_1*a_{12}+y_2*a_{22}+y_3*a_{32} \geq p_2$

and $y_1, y_2, y_3 \geq 0$

We only need the units of $\underline{y}$ at this point. Lets look at the units of the constraining inequalities of the dual program:

$[y_1]$ * (dkg/piece) + $[y_2]$ * (l/piece) + $[y_3]$ * (g/piece) ≥ Ft/piece

So, in case of $\underline{y}$ the units must be: $[y_1]$= Ft/dkg; $[y_2]$= Ft/l; $[y_3]$= Ft/g

Therefore, [Objective function in dual program]=

=(Ft/dkg) *(dkg) + (Ft/l) *(l) + (Ft/g) *(g) = Ft (of course the same as in case of primal)

Primal and Dual Programs, practical example

The dual program:

Minimize $c_1y_1 + c_2y_2 + c_3y_3$

Subject to

$y_1 * a_{11} + y_2 * a_{21} + y_3 * a_{31} \geq p_1$

$y_1 * a_{12} + y_2 * a_{22} + y_3 * a_{32} \geq p_2$

and $y_1, y_2, y_3 \geq 0$

→ *substituting the values*

Minimize $200y_1 + 100y_2 + 500y_3$

Subject to

$2y_1 + y_2 + 3y_3 \geq 5$

$2y_1 + 2y_2 + y_3 \geq 6$

and $y_1, y_2, y_3 \geq 0$

If we solve this problem, we get the following solution:

y_1 = 1,222 Ft/dkg; y_2 = 1,622 Ft/l; y_3 = 0,311 Ft/g; minimum of obj.func= 500 Ft.

These are the shadow prices (the optimal values of dual variables) .

Conversion of Primal Program of the WYNDOR GLASS Co problem to Dual Program

Primal Problem in Algebraic Form

Maximize $Z = 3x_1 + 5x_2,$

subject to

$$x_1 \le 4$$
$$2x_2 \le 12$$
$$3x_1 + 2x_2 \le 18$$

and $x_1 \ge 0, \quad x_2 \ge 0.$

Dual Problem in Algebraic Form

Minimize $W = 4y_1 + 12y_2 + 18y_3,$

subject to

$$y_1 + 3y_3 \ge 3$$
$$2y_2 + 2y_3 \ge 5$$

and

$y_1 \ge 0, \quad y_2 \ge 0, \quad y_3 \ge 0.$

Primal and Dual Programs, practical example2

This is the defined primal task. Determine and solve the dual of it in Excel!

Max $Z=4x_1 + 5x_2 + x_3$

Subject to

$3x_1+2x_2 \leq 10$

$x_1+4x_2 \leq 11$

$3x_1+3x_2+x_3 \leq 13$

and $x_1, x_2, x_3 \geq 0$

BUDAPEST UNIVERSITY OF TECHNOLOGY AND ECONOMICS

Dr. Tibor SIPOS Ph.D.
Anteneh AFEWORK
Zsombor SZABÓ

email: sipos.tibor@mail.bme.hu

MŰEGYETEM 1782